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Abstract

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MATHEMATICS

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ON THE d -CHARACTERISTIC OF A LINEAR OPERATOR

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Let E and F be vector spaces over one and the same field, and let $\mathcal{L}(E, F)$ be the set of all possible linear operators mapping E into F . Denote by $\mathfrak{Z}_A$ the set of all zeros of the operator $A \in \mathcal{L}(E, F)$. The ordered pair (α_A, β_A) , where $\alpha_A = \dim \mathfrak{Z}_A$, $\beta_A = \text{codim}_F A(E)$, is called the d -characteristic of the operator A .

In the present note we shall be interested in the dependence between the d -characteristics of two operators from $\mathcal{L}(E, F)$ connected by certain conditions. We first consider some auxiliary decompositions of the spaces E and F and note a number of properties of these decompositions.

1°. Let P_1 and P_2 be the projection operators generated by the decomposition of E into the direct sum of subspaces E_1 and E_2 ($E = E_1 \oplus E_2$, $P_1(E) = E_1$, $P_2(E) = E_2$), and let E_0 be some subspace in E . Put $M_1 = E_0 \cap E_1$, $M_2 = E_0 \cap E_2$, and take some subspace M_0 complementary to $M_1 \oplus M_2$ in E_0 ($M_0 = \text{comp}_{E_0}(M_1 \oplus M_2)$). Let $M_3 = P_1(M_0)$, $M_4 = P_2(M_0)$. Then $M_i \cap M_j = \{0\}$ for $i \neq j$ ($i, j = 0, 1, 2, 3, 4$), and the spaces M_0, M_3, M_4 are isomorphic to one another.

2°. If $M_5 = \text{comp}_{E_1}(M_1 \oplus M_3)$, $M_6 = \text{comp}_{E_2}(M_2 \oplus M_4)$, then

$$E = E_0 \oplus M_4 \oplus M_5 \oplus M_6 (= E_0 \oplus M_3 \oplus M_5 \oplus M_6).$$

Indeed, for any $x \in E$ we shall have

$$x = \sum_{i=1}^6 x_i, \quad \text{where } x_i \in M_i.$$

But $x_3 = P_1 x_0 = x_0 - P_2 x_0$, where $x_0 \in M_0$; consequently,

$$x = (x_0 + x_1 + x_2) + (x_4 - P_2 x_0) + x_5 + x_6,$$

i.e. $x \in E_0 \oplus M_4 \oplus M_5 \oplus M_6$.

3°. If $T \in \mathcal{L}(E, F)$, $E_0 = \mathfrak{Z}_T$, then: a) $T(M_3) = T(M_4)$; b)

$$T(E) = T(M_3) \oplus T(M_5) \oplus T(M_6) \quad (= T(E_1) \oplus T(M_6));$$

c)

$$T(E_1) \cap T(E_2) = T(M_3).$$

a) Let $x_3 \in M_3$. Then $x_3 = P_1 x_0 = x_0 - P_2 x_0 = x_0 - x_4$, where $x_0 \in M_0$, $x_4 \in M_4$. Hence $Tx_3 = -Tx_4 \in T(M_4)$, $T(M_3) \subset T(M_4)$. Similarly we obtain the inclusion $T(M_4) \subset T(M_3)$.

b) Since $E = M_1 \oplus \dots \oplus M_6$, $T(M_1) = T(M_2) = \{0\}$, $T(M_3) = T(M_4)$, it follows that

$$T(E) = T(M_3) + T(M_5) + T(M_6).$$

We prove that the spaces $T(M_3)$, $T(M_5)$, $T(M_6)$ are pairwise disjoint, whence it will follow that $T(E)$ is their direct sum. Let us carry out the proof, for example, for $T(M_5)$ and $T(M_6)$. Let $y \in T(M_5) \cap T(M_6)$; then $y = Tx_5 = Tx_6$, $x_5 - x_6 = x_0 + x_1 + x_2$, where $x_i \in M_i$, $i = 0, 1, 2, 5, 6$. But

$$x_0 = P_1 x_0 + P_2 x_0 = x_3 + x_4;$$

therefore,

$$x_1 + x_2 + x_3 + x_4 - x_5 + x_6 = 0.$$

Hence $x_i = 0$, $i = 1, \dots, 6$, and $y = 0$.

c) The equality $T(E_1) \cap T(E_2) = T(M_3)$ follows from a), b) and the decompositions

$$T(E_1) = T(M_3) \oplus T(M_5), \quad T(E_2) = T(M_4) \oplus T(M_6).$$

In items 1 and 2 some constructions were performed in the space E . In what follows we shall need the same constructions in the space F . The corresponding notation is obtained by replacing E_i by F_i ($i = 0, 1, 2$), M_i by N_i ($i = 1, \dots, 6$), and P_i by Q_i ($i = 1, 2$).

Let operators A and T from $\mathcal{L}(E, F)$ be given. With their aid one can carry out the constructions indicated above in the spaces E and F , putting $E_2 = \mathfrak{Z}_A$, $E_0 = \mathfrak{Z}_T$, $F_1 = A(E)$, $F_0 = T(E)$, and taking as E_1 and F_2 some complements to $\mathfrak{Z}_A$ and $A(E)$, respectively, in E and F . In what follows we shall always assume that the constructions under consideration in E and F have been carried out precisely in this way.

Theorem 1. *If $\mathfrak{Z}_{Q_1 TP_1} = \mathfrak{Z}_A$ and $Q_1 TP_1(E) = A(E)$, then the following isomorphisms hold: 1) $\mathfrak{Z}_A \sim \mathfrak{Z}_T \oplus M_6$; 2) $\text{coker}_F A(E) \sim \text{coker}_F T(E) \oplus N_2$; 3) $M_6 \sim N_2$.*

Proof. 1) From the condition $\mathfrak{Z}_{Q_1 TP_1} = \mathfrak{Z}_A$ it follows that $\mathfrak{Z}_{TP_1} = \mathfrak{Z}_A$, whence $M_1 = \{\theta\}$. Therefore, by item 1°, $\mathfrak{Z}_T \sim M_2 \oplus M_4$. But $\mathfrak{Z}_A = M_2 \oplus M_4 \oplus M_6$, and hence $\mathfrak{Z}_A \sim \mathfrak{Z}_T \oplus M_6$.

- 2) From the condition $Q_1TP_1(E) = A(E)$ it follows that $Q_1T(E) = A(E)$, whence, by item 1⁰ (as applied to F), $N_5 = \{\theta\}$. Hence, by item 2⁰ (for F), $\text{coker}_F T(E) = N_4 \oplus N_6$. But $\text{coker}_F A(E) = N_2 \oplus N_4 \oplus N_6$, and therefore $\text{coker}_F A(E) = \text{coker}_F T(E) \oplus N_2$.
- 3) We show that $T(E) = TP_1(E) \oplus N_2$. Indeed, from the condition $\mathfrak{Z}_{Q_1TP_1} = \mathfrak{Z}_A$ it follows that $TP_1(E) \cap N_2 = \{\theta\}$. Moreover, if $y \in T(E)$, then, putting $Q_1y = z$ ($\in A(E)$), we find (taking into account the condition $Q_1TP_1(E) = A(E)$) an element $y_1 \in TP_1(E)$ such that $Q_1y_1 = z$. Then $Q_1(y - y_1) = \theta$, i.e. $y_2 = y - y_1 \in N_2$. Thus $y = y_1 + y_2$, where $y_1 \in TP_1(E)$, $y_2 \in N_2$.

Comparing the decomposition just found, $T(E) = T(E_1) \oplus N_2$, with the decomposition $T(E) = T(E_1) \oplus T(M_6)$ (item 3⁰, b), we conclude that $N_2 \sim T(M_6)$. But $\mathfrak{Z}_T \cap M_6 = \{\theta\}$, and therefore $N_2 \sim M_6$.

We note that assertion 1) of the theorem has been proved only with the aid of the first hypothesis in the weakened form ($\mathfrak{Z}_{TP_1} = \mathfrak{Z}_A$), and assertion 2) only with the aid of the second hypothesis, also in the weakened form ($Q_1T(E) = A(E)$). Both hypotheses are used only in the proof of assertion 3).

Theorem 2. *Suppose that both hypotheses of Theorem 1 are satisfied and at least one of the numbers α_A or β_A is finite. Then the spaces M_6 and N_2 are finite-dimensional, and the indices of the operators A and T are equal:*

$$\alpha_A - \beta_A = \alpha_T - \beta_T.$$

Proof. Since $M_6 \subset \mathfrak{Z}_A$ and $N_2 \subset \text{coker}_F A(E)$, on the basis of assertion 3) of Theorem 1 we conclude that M_6 and N_2 are finite-dimensional. Therefore, by assertions 1) and 2) of Theorem 1, we can write

$$\dim \mathfrak{Z}_A = \dim \mathfrak{Z}_T + \dim M_6, \quad \text{codim}_F A(E) = \text{codim}_F T(E) + \dim N_2.$$

But $\dim M_6 = \dim N_2$, whence $\dim \mathfrak{Z}_A - \text{codim}_F A(E) = \dim \mathfrak{Z}_T - \text{codim}_F T(E)$, i.e. $\alpha_A - \beta_A = \alpha_T - \beta_T$.

From assertions 1) and 2) of Theorem 1 it follows that $\alpha_T \leq \alpha_A$ and $\beta_T \leq \beta_A$. If, however, the conditions of Theorem 2 are satisfied and α_A is finite, then $\alpha_T = \alpha_A$; analogously, if β_A is finite, then $\beta_T = \beta_A$.

Up to this point all the considerations have been carried out for linear operators in vector spaces without topology. Next we shall apply the results obtained to linear operators in Banach and Hilbert spaces.

Let X and F be Banach spaces, A a linear closed operator with domain $E \subset X$ and range $A(E) \subset F$, closed in F . Take the operator $T = A + B$, where B is a linear bounded operator defined on E .

Theorem 3. *If the operators P_2 and Q_1 , projecting E onto $\mathfrak{Z}_A$ and F onto $A(E)$, are bounded, then there exists an $\varepsilon > 0$ such that*

$$\mathfrak{Z}_{Q_1TP_1} = \mathfrak{Z}_A$$

and

$$Q_1TP_1(E) = A(E),$$

as soon as $\|B\| < \varepsilon$.

Proof. Let A_1 and T_1 be the restrictions of the operators A and T to E_1 . We show that, for sufficiently small values of $\|B\|$, the boundedness of the operator P_2 implies the continuous invertibility of T_1 (simple invertibility of T_1 would mean that $\mathfrak{Z}_{TP_1} = \mathfrak{Z}_A$). Indeed, from the properties of the operator A it follows —

guarantees the existence of such an $m > 0$ that for every $y \in A(E)$ there is its preimage $x \in E$ ($Ax = y$), satisfying the inequality $\|x\| \leq m\|y\|$. Putting $x_1 = P_1x$, we shall have $\|x_1\| \leq \|P_1\|\|x\| \leq \|P_1\|m\|y\| = m_1\|y\|$ ($m_1 > 0$). Since y was taken arbitrarily in $A(E) = A_1(E_1)$ and $A_1x_1 = y$, this means that A_1 is continuously invertible. Now take any element $z \in T_1(E_1)$. Then $z = Ax_1 + Bx_1$ ($x_1 \in E_1$),

$$\|z\| \geq \|Ax_1\| - \|Bx_1\| \geq (1/m_1 - \|B\|)\|x_1\|,$$

whence it is clear that if $\|B\| < 1/m_1 = \varepsilon_1$, then T_1 is continuously invertible.

Let us further note that, since $\mathfrak{Z}_{TP_1} = \mathfrak{Z}_A$ and $\mathfrak{Z}_{Q_1} = \text{codim}_F A(E)$, in order to prove the equality $\mathfrak{Z}_{Q_1TP_1} = \mathfrak{Z}_A$ it suffices to establish that $T_1(E_1) \cap \text{codim}_F A(E) = \{0\}$. First of all, from the boundedness of Q_2 we conclude that

$$\|y_2 - y_1\| \geq \frac{1}{\|Q_2\|}\|y_2\|$$

for any $y_1 \in A(E)$ and $y_2 \in \text{codim}_F A(E)$. Hence, for any $z \in T_1(E_1)$, we shall have

$$\begin{aligned} \|y_2 - z\| &= \|y_2 - Ax_1 - Bx_1\| \geq \\ &\geq \|y_2 - Ax_1\| - \|Bx_1\| \geq \frac{1}{\|Q_2\|}\|y_2\| - \|B\|\|x_1\| \\ &\geq \frac{1}{\|Q_2\|}\|y_2\| - \frac{\|B\|}{\varepsilon_1 - \|B\|}\|z\|, \end{aligned}$$

where $x_1 = T_1^{-1}z$. This shows that, for a sufficiently small value of $\|B\|$ ($\|B\| < \varepsilon_2$), the elements y_2 and z cannot coincide if they are different from zero, i.e. $T_1(E_1) \cap \text{codim}_F A(E) = \{0\}$.

To prove the equality $Q_1TP_1(E) = A(E)$, write the operator Q_1TP_1 in the form

$$Q_1TP_1 = Q_1(A + B)P_1 = A + Q_1BP_1.$$

Since the operator Q_1BP_1 is small when B is small ($\|B\| < \varepsilon_3$) and $Q_1BP_1(E) \subset A(E)$, the required equality is a consequence of the stability of the epimorphism property of a closed operator.

The ε indicated in the theorem is equal to $\min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

If X and F are Hilbert spaces, then, by virtue of the closedness of $\mathfrak{Z}_A$ and $A(E)$, the existence of bounded operators P_2 and Q_1 is ensured. Therefore, for Hilbert spaces the theorem on stability of the index (i.e. the fact that, if at least one of the numbers α_A or β_A is finite, then the addition of a small operator B to the operator A does not change the index) follows completely from our preceding theorems. For Banach spaces, bounded operators P_2 and Q_1 , in any case, exist if both numbers α_A and β_A are finite, and in this case the stability of the index still follows from the same theorems. However, as is known, for Banach spaces the stability property of the index also holds when only one of the numbers α_A or β_A is finite (see, for example, I. Ts. Gohberg and M. G. Krein (1), where results concerning the d -characteristic of operators in a Banach space are set out in detail), but this fact does not follow from the theorems of the present note.

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CITED LITERATURE

1. I. Ts. Gohberg, M. G. Krein, UMN, 12, no. 2 (74), 43 (1957).

Note: Figure translations are in progress. See original paper for figures.

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