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Abstract

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MATHEMATICS

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ON ALMOST-PERIODIC SOLUTIONS OF NONLINEAR SYSTEMS

(Presented by Academician I. G. Petrovskii, May 27, 1965)

In the present note sufficient conditions are indicated in order that the module of Fourier exponents of an almost-periodic solution of a nonlinear almost-periodic system be contained in the module of the right-hand side. In addition, theorems are formulated on the almost-periodicity of bounded solutions of almost-periodic "monotone" systems (see Definition 3).

1. Let an n -dimensional vector system be given

$$\dot{x} = F(x, t), \quad (1)$$

where $F(x, t)$ satisfies the local theorems of existence and uniqueness. Suppose that $F(x, t)$ is almost-periodic in t , uniformly in the cylinder $C_\Gamma\{x \in \Gamma, t \in I\}$, with module of Fourier exponents M_Γ (here Γ is some closed, bounded set in R^n , and $I = (-\infty, +\infty)$).

We shall say that $x(t)$ is contained in the cylinder C_Γ if $x(t) \in \Gamma, t \in I$. Let system (1) have an almost-periodic solution $x(t)$ contained in C_Γ . We give some conditions sufficient for the module of Fourier exponents M of this solution to satisfy the inclusion

$$M \subseteq M_\Gamma. \quad (2)$$

Denote by C the space of all continuous n -dimensional vector-functions bounded on I .

Definition 1. A set $K \subset C$ will be called an **invariant class with respect to translations** if, from the fact that $x(t) \in K, h \in I$, it follows that $x(t+h) \in K$, and from the fact that $x(t+h_n) \rightarrow y(t)$ (uniformly on I), it follows that $y(t) \in K$. Obviously, the intersection of two invariant classes is again an invariant class.

Theorem 1. Let $x(t)$ be some almost-periodic solution of system (1) in C_Γ . If to any other solution $y(t)$ of this system contained in C_Γ one can assign some

invariant class K (depending, generally speaking, on $y(t)$) in such a way that $x(t) \in K$ and $y(t) \in K$, then (2) is valid for $x(t)$.

Hence, in particular, we obtain:

Corollary 1. If the almost-periodic solution $x(t)$ is the only representative in C_Γ of some invariant class K , then (2) is valid for $x(t)$.

If here, as K , one takes the set K^n of all almost-periodic n -vector-functions, then we obtain:

Corollary 2. If $x(t)$ is the unique almost-periodic solution of system (1) contained in C_Γ (other solutions extendable in both directions, but no longer almost-periodic, may be contained in C_Γ), then (2) holds for $x(t)$. In particular, if the almost-periodic solution $x(t)$ is in fact the unique solution wholly contained in C_Γ , then (2) holds for it.

Corollary 3. If (1) is a system with almost-periodic convergence ⁽²⁾, then the "limiting" solution of this system satisfies the inclusion (2).

Along with system (1), let us consider the entire collection of systems

$$\dot{x} = L(x, t), \quad (3)$$

where $L(x, t) \in H\{F(x, t)\}$ ⁽³⁾, i.e. $L(x, t)$ is obtained by a uniform limiting transition in C_Γ from some sequence of translates $\{F(x, t + h_n)\}$.

We now give some definitions and a theorem due to Amerio.

Definition 2. A solution $x^*(t)$ contained in C_Γ is called **separated in C_Γ** if for any other solution $y(t)$ contained in C_Γ , for all $t \in I$ we have

$$\|x^*(t) - y(t)\| \geq \rho^* > 0,$$

where ρ^* depends only on $x^*(t)$. If $x^*(t)$ is the unique solution contained in C_Γ , then we shall also call it separated in C_Γ .

If, for any system (3), all solutions contained in C_Γ are separated, then there can be only a finite number l of them, independent of $L(x, t)$ from the set $H\{F(x, t)\}$.

Amerio's theorem ⁽¹⁾. *If all solutions contained in C_Γ of any system from (3) are separated, then they are all almost periodic.*

In particular, if any system (3) admits only one solution contained in C_Γ , then it is almost periodic.

Under the conditions of this theorem, in the general case one can say nothing about the relation between the module M_Γ of the right-hand side of system (1) and the modules M of the separated solutions. With the aid of Corollary 2 and Theorem 1, where as invariant classes we take the sets

$$K_1 = K_{\text{ap}}^{(n)} \cap K_d^{(n)}; \quad K_2 = K_{\text{ap}}^{(n)} \cap K_s^{(n)}; \quad K_3 = K_{\text{ap}}^{(n)} \cap K_i^{(n)}, \quad (4)$$

we indicate a condition sufficient for all separated solutions in Amerio's theorem to satisfy (2). Here $K_{\text{ap}}^{(n)}$ is the set of all almost-periodic n -vector functions; K_d^n is the set of all continuous, bounded n -vector functions with a given diameter d

$$d = \sup_{t_1, t_2 \in I} \|x(t_1) - x(t_2)\|; \quad (5)$$

K_s^n is the set with given s

$$s = \sup_{t \in I} \|x(t)\|; \quad (6)$$

$K_i^{(n)}$ is the set with given i

$$i = \inf_{t \in I} \|x(t)\|. \quad (7)$$

Corollary 4. *If in Amerio's theorem the number of separated solutions is $l = 1$, then (2) is valid for the separated solution.*

In the general case, if for each pair chosen from among the $l > 1$ separated solutions there is at least one of the characteristics (5), (6), or (7) which takes different numerical values on the two solutions of the chosen pair, then all l separated solutions satisfy the inclusion (2).

If in Theorem 1 and in all the corollaries the function $F(x, t)$ is considered "purely" periodic with least period T , then in all the stated assertions the almost-periodic solution $x(t)$ will be periodic with least—

with the smaller period $\bar{T} = T/q$, where q is some natural number ($q \geq 1$), i.e. a Massera harmonic ⁽⁴⁾.

From Corollary 1 and the definition of the invariant classes $K_1^{(n)}$, $K_2^{(n)}$, and $K_3^{(n)}$ we obtain that if system (1) in C_Γ has some almost-periodic solution $x(t)$ with module M , not contained in M_Γ , then in C_Γ there must be at least one more almost-periodic solution $y(t)$ with the same module M such that any of the characteristics (5), (6), and (7) of this solution is equal to the corresponding characteristic of the solution $x(t)$, i.e. solutions not satisfying relation (2) cannot exist "in isolation."

In the general case of n -dimensional systems, in order for the inclusion (2) to hold we had to impose certain "uniqueness" conditions in some invariant class. It turns out that in the case $n = 1$ no additional conditions are required; namely, the following assertion holds.

Theorem 2. *If the first-order equation*

$$\dot{x} = f(x, t), \quad (8)$$

where $f(x, t)$ satisfies the local existence and uniqueness theorems and is an almost-periodic function uniformly in the strip $C_\Gamma\{a \leq x \leq b, t \in I\}$, has in this strip an almost-periodic solution $x(t)$, then the module of its Fourier exponents is necessarily contained in the module M_Γ of the function $f(x, t)$.

For $n > 1$ Theorem 2 is false, since it is known ⁽⁴⁾ that already for $n = 2$ a periodic system may have subharmonics.

Corollary 5. *If in equation (8) $f(x, t)$ is a periodic function with least period T , then every almost-periodic solution is a harmonic.*

This assertion, due to Massera, is a simple consequence of Theorem 2.

2. Let a Lyapunov function $V(x) \in C^{(1)}(R^{(n)})$ be given, positive definite and unboundedly increasing at infinity, i.e. for every $A > 0$ there exists $B > 0$ such that

$$V(x) > A, \quad \text{if } \|x\| > B.$$

Suppose, moreover, that we have a vector function $F(x, t)$, continuous in the aggregate of its variables.

Definition 3. We shall say that $F(x, t)$ is a V -monotonically decreasing function with respect to x in C_Γ if for every pair of vectors x and $y \in \Gamma$ ($x \neq y$), for all $t \in I$,

$$W(x, y, t) \equiv (\text{grad } V(x - y), F(x, t) - F(y, t)) < 0. \quad (9)$$

V -monotone increase is defined analogously, with the sign of the inequality changed to the opposite. In what follows we shall consider V -monotone decrease and shall simply say V -monotonicity. For monotone increase everything is rephrased in the obvious way.

Theorem 3. *If in system (1) the function $F(x, t)$ is V -monotone with respect to x in C_Γ and almost-periodic with respect to t uniformly on the set Γ , then in the cylinder C_Γ there can be no more than one solution defined on all of I (a limiting solution). This solution is an almost-periodic function with module of Fourier exponents M contained in the module M_Γ of the function $F(x, t)$. If some other solution remains in C_Γ for all $t \geq t_0$, then it asymptotically approaches the limiting solution as $t \rightarrow +\infty$.*

Theorem 4. *Suppose that in system (1) the function $F(x, t)$ is V -monotone in the whole space R^n and almost-periodic with respect to t uniformly in each set $D_R\{V(x) \leq R\}$, with a module of Fourier exponents uniform in D_R*

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M_R . If system (1) has, in some cylinder $C_R\{x \in D_R, t \in I\}$, a solution $x(t)$ bounded on the entire axis, then this solution is almost-periodic with the Fourier exponent module M , contained in M_R . All other solutions are unbounded to the left and tend asymptotically to $x(t)$ as $t \rightarrow +\infty$, i.e., almost-periodic convergence (2) takes place.

For $V(x) = \frac{1}{2}\|x\|^2$, this theorem generalizes Opial's theorem (5) to the case of n dimensions.

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