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Abstract

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MATHEMATICS

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TENSOR PRODUCTS AND FUNCTORS IN CATEGORIES OF BANACH SPACES DEFINED BY KB -LINEALS

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In the author's note ⁽¹⁾, for each KB -lineal E a functor Φ_E was constructed in the category of Banach spaces, which is a broad and natural generalization of the functors defined by minimal normed ideals of sequences and measurable functions ⁽²⁾. The functor Φ_E assigns to a Banach space X the completion of the algebraic tensor product $E \otimes X$ with respect to the crossnorm n_E , defined by the formula

$$n_E \left(\sum_{k=1}^n e_k \otimes x^k \right) = \\ = \inf \left\{ \|u\|_E : u \geq \left| \sum_{k=1}^n e_k \langle x_k, x' \rangle \right| \text{ for all } x' \in X', \|x'\| \leq 1 \right\};$$

this completion will henceforth be denoted by $E \widetilde{\otimes} X$. In ⁽¹⁾ the properties of the functor Φ_E were studied; in particular, the space of mappings $\{\Phi_E \rightarrow \Phi_F\}$ of one such functor into another and the dual functor $D\Phi_E$ were described.

In the present note, which is a continuation of ⁽¹⁾, further properties of the tensor product $E \widetilde{\otimes} X$ and of the functor Φ_E are reported. For definitions relating to functors in categories of Banach spaces, see ⁽²⁾.

An L -space is called, following S. Kakutani ⁽³⁾, a KB -lineal E satisfying the condition $\|e_1 + e_2\| = \|e_1\| + \|e_2\|$ for any positive $e_1, e_2 \in E$; in ⁽³⁾ it is proved that every L -space is linearly isometric and structurally isomorphic to the KB -lineal L_M^1 , where $M = (S, \Sigma, \mu)$ is some measure space.

An M -space is called, following S. Kakutani ⁽⁴⁾, a KB -lineal E satisfying the condition $\|\sup(e_1, e_2)\| = \max(\|e_1\|, \|e_2\|)$ for any positive $e_1, e_2 \in E$; in ⁽⁴⁾ it is

proved that every M -space is linearly isometric and structurally isomorphic to a sublineal of a special type of the KB -lineal C_Q , where Q is some bicompactum.

Theorem 1. Let E be a KB -lineal.

- 1) In order that, for every X , the crossnorm n_E coincide on $E \otimes X$ with the maximal crossnorm π , it is necessary and sufficient that E be an L -space.
- 2) In order that, for every X , the crossnorm n_E coincide on $E \otimes X$ with the minimal crossnorm ε , it is necessary and sufficient that E be an M -space.

We shall need some definitions from note ⁽¹⁾; let us recall them. We say that for a KB -lineal E condition $(*)$ is fulfilled if, for every monotonically increasing sequence bounded above of elements $e_n \geq 0$ of E , $\|e_n\| \rightarrow \infty$ as $n \rightarrow \infty$; condition $(**)$, if for every monotonically increasing sequence of elements $e_n \geq 0$ of E having a supremum $e = \sup_n e_n$, $\|e_n\| \rightarrow \|e\|$ as $n \rightarrow \infty$; condition $(***)$, if for every Banach space X and every

for $z \in E \otimes X$

$$n_E(z) = \inf \left\| \sum_{k=1}^n |e_k| \|x_k\| \right\|_E,$$

where the lower bound is taken over all possible representations

$$z = \sum_{k=1}^n e_k \otimes x_k.$$

Condition $(*)$ is satisfied by the majority of known KB -lineals, in particular by all KB -spaces ^(5,6); it is not satisfied by the KB -lineal C_0 . Conditions $(**)$ and $(***)$ are fulfilled in all known cases.

We call a mapping $\alpha : E \rightarrow Y$ of a KB -lineal E into a Banach space Y **summing** if it carries every series $\sum_{k=1}^{\infty} e_k$ from E for which the series $\sum_{k=1}^{\infty} |e_k|$ converges into an absolutely convergent series $\sum_{k=1}^{\infty} \alpha e_k$ in Y .

We call a mapping $\alpha : X \rightarrow E$ of a Banach space X into a KB -lineal E **proper** if it carries the unit ball of X into a subset of E that is bounded in the sense of the ordering.

Theorem 2. Let E be a KB -lineal satisfying condition $(***)$, and let X be a Banach space. Then the space $(E \widetilde{\otimes} X)'$, conjugate to $E \oplus X$, is isometric to the space of summing mappings of E into X' , endowed with the norm

$$\nu_E(\alpha) = \sup \left[\sum_{k=1}^n \|\alpha e_k\|_{X'} / \left\| \sum_{k=1}^n |e_k| \right\|_E \right],$$

where the least upper bound is taken over all possible finite collections of elements $e_1, \dots, e_n \in E$.

The space E' , conjugate to the KB -lineal E , is a conditionally complete KB -lineal with respect to the natural order (see ⁽⁵⁾ or ⁽⁶⁾).

Proposition 1. Let E be a KB -lineal, X a Banach space, and let $\alpha : X \rightarrow E$ be a proper mapping. Then its conjugate $\alpha^* : E' \rightarrow X'$ is a summing mapping and

$$\nu_{E'}(\alpha^*) \leq n(\alpha) = \inf\{\|u\|_E : u \geq |\alpha x| \text{ for all } x \in X, \|x\| \leq 1\}.$$

Proposition 2. Let E be a KB -lineal, Y a Banach space, and X its closed subspace. Then $E \widetilde{\otimes} X$ is a closed subspace of $E \widetilde{\otimes} Y$.

Proposition 3. Let E be a KB -lineal satisfying condition $(***)$, X a Banach space, X_0 its closed subspace, and $Y = X/X_0$. Then $E \widetilde{\otimes} Y$ is isometric to the quotient space

$$E \widetilde{\otimes} X / E \widetilde{\otimes} X_0.$$

Let E be a KB -lineal and X a Banach space. The identity mapping of $E \widetilde{\otimes} X$ onto itself extends to a continuous linear mapping

$$\omega : \Phi_E(X) \equiv \widetilde{E \otimes X} \rightarrow E \widehat{\otimes} X^*.$$

We say that the functor Φ_E satisfies the **condition of one-to-one correspondence** if the mapping ω is one-to-one for every X in the category of Banach spaces under consideration.

Theorem 3. Let the KB -lineal E satisfy condition $(***)$, and let Φ_E satisfy the condition of one-to-one correspondence. Then the functor Φ_E is exact, i.e., every exact triple

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$$

is carried by it into the exact triple

$$\Phi_E(X) \xrightarrow{\Phi_E(\alpha)} \Phi_E(Y) \xrightarrow{\Phi_E(\beta)} \Phi_E(Z).$$

In ⁽¹⁾ a realization of the space $\{\Phi_E \rightarrow \Phi_F\}$ was described in the form of the space $(E \rightarrow F)_r$ of regular mappings of E into F . The space $\{\Phi_E \rightarrow \Phi_F\}$ admits a somewhat different description under other conditions on the KB -lineals E and F .

* $\widetilde{E \otimes X}$ denotes, as usual, the completion of $E \otimes X$ with respect to the minimal crossnorm.

Denote by c_0^n the n -dimensional coordinate space of elements $x = (\xi_1, \dots, \xi_n)$ with norm $\|x\|_{c_0^n} = \max_{1 \leq k \leq n} |\xi_k|$. Let E, F be KB -linear spaces. A mapping $\alpha \in (E \rightarrow F)$ will be called K -bounded if it sends every sequence (e_k) bounded in the sense of the ordering in E into a sequence (αe_k) of the same kind in F .

We shall say that a KB -linear space E has property (N) if, for every monotonically increasing and bounded above sequence of elements $e_n \geq 0$ ($n = 1, 2, \dots$) from E and every number $\delta > 0$, there exists an element $e_\delta \in E$ majorizing all the e_n and such that

$$\|e_\delta\| \leq \lim_{n \rightarrow \infty} \|e_n\| + \delta.$$

For a conditionally σ -complete KB -linear space, conditions (N) and () are equivalent; in the general case (N) implies ().

Theorem 4. Let E, F be KB -linear spaces satisfying conditions () and (N); let $\mathcal{K}$ be a category of Banach spaces containing spaces X_n for which there exist isometric embeddings $i_n : c_0^n \rightarrow X_n$ ($n = 1, 2, \dots$). Then, for the functors Φ_E and Φ_F considered in $\mathcal{K}$, the space $\{\Phi_E \rightarrow \Phi_F\}$ is isometric to the space of K -bounded mappings $\alpha : E \rightarrow F$, endowed with the norm*

$$\|\alpha\|_0 = \sup \left[\left\| \sup_{1 \leq k \leq n} |\alpha e_k| \right\|_F / \left\| \sup_{1 \leq k \leq n} |e_k| \right\|_E \right],$$

where the supremum is taken over all finite sets of elements $e_1, \dots, e_n \in E$.

Remark. Theorem 3 in ⁽¹⁾ is formulated with a superfluous restriction on the category $\mathcal{K}$; for its validity it is enough to impose on $\mathcal{K}$ the same condition as in Theorem 4 of the present paper. Similarly, Theorem 4 in ⁽¹⁾ holds if one requires only that $\mathcal{K}$ contain spaces X_n for which there exist isometric embeddings $i_n : l_n^1 \rightarrow X_n$, where l_n^1 is the n -dimensional coordinate space of elements

$$x = (\xi_1, \dots, \xi_n) \quad \text{with norm} \quad \|x\|_{l_n^1} = \sum_{k=1}^n |\xi_k|.$$

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Note: Figure translations are in progress. See original paper for figures.

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