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Abstract

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SELF-SIMILAR MOTIONS OF A RELATIVISTIC GAS IN GENERAL RELATIVITY IN THE CASE OF POINT SYMMETRY

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For a number of problems in relativistic astrophysics and cosmology, it is of interest to consider all possible self-similar motions of a relativistic gas in general relativity (g.r.) possessing point symmetry.

All parameters of the moving gas, for a given equation of state, together with the metric, which in the present case has the form ⁽¹⁾

$$ds^2 = e^\nu c^2 dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \quad (1)$$

(where ν and λ are functions of r and t), are described by the Einstein equations $R_i^k - \frac{1}{2}\delta_i^k R = \chi T_i^k$ and by the equations of motion following from them

$$T_{i;k}^k = 0. \quad (2)$$

A general investigation of self-similarity of the indicated system of equations is carried out by a method analogous to that used for finding self-similar solutions for gas motion in special relativity (s.r.) ^(2, 3).

From system (2), for the adiabatic motion of a gas we have

$$\begin{aligned} \frac{d(wu_i)}{ds} + \frac{\partial w}{\partial x^i} &= \frac{w}{2} u^k u^l \frac{\partial g_{kl}}{\partial x^i} + T \frac{\partial \sigma}{\partial x^i}, \\ \frac{\partial}{\partial x^k} \left(\frac{u^k}{V} \sqrt{-g} \right) &= 0, \quad \frac{d\sigma}{ds} = 0. \end{aligned} \quad (3)$$

Here u^i is the 4-velocity; w and σ are the heat content and entropy, which can be expressed in terms of the pressure P and the specific volume V ,

$$w = \frac{k}{k-1}PV + \alpha c^2, \quad PV^k = \sigma. \quad (4)$$

It is useful next to write system (3) in a form similar to the system of equations for the adiabatic motion of a gas in s.r.:

$$\left(\frac{\partial \ln w}{\partial r} + \frac{a}{c^2} \frac{\partial \ln w}{\partial \tau_0} \right) + \frac{1}{\theta^2 c^2} \left(\frac{\partial a}{\partial \tau_0} + a \frac{\partial a}{\partial r} \right) = -\frac{1}{2} \left(\frac{\partial \nu}{\partial r} + \frac{a}{c^2} \frac{\partial \lambda}{\partial \tau_0} \right) + \frac{T\theta^2}{w} \frac{\partial \sigma}{\partial r} - \left(\frac{\partial \ln V}{\partial \tau_0} + a \frac{\partial \ln V}{\partial r} \right) + \frac{1}{\theta^2} \left(\frac{\partial a}{\partial r} + \frac{a}{c^2} \frac{\partial a}{\partial \tau_0} \right)$$

$$\frac{\partial \sigma}{\partial \tau_0} + a \frac{\partial \sigma}{\partial r} = 0.$$

In system (5), a is the root-mean-square velocity measured with respect to proper time; $d\tau = \sqrt{-g_{00}} dt$, $\theta^2 = 1 - a^2/c^2$; further, to shorten the notation an auxiliary time $d\tau_0 = e^{(\lambda-\nu)/2} dt$ is introduced. In doing this one should remember that the quantities τ_0 and r are not independent.

System (5) differs from the equations of gas motion in special relativity by the presence of two functions λ and ν , describing the gravitational forces acting on the gas in its own gravitational field. To determine them one should use Einstein's gravitational equations, which for the centrally symmetric motion of a gas with metric (1) have the form given in (1). From the indicated system of equations one can obtain the relations

$$\frac{\partial \lambda}{\partial \tau_0} \left(1 + \frac{a^2}{c^2} \right) + a \left(\frac{\partial \nu}{\partial r} + \frac{\partial \lambda}{\partial r} \right) = 0, \quad \frac{\partial \lambda}{\partial \tau_0} = -\frac{\chi(P + \varepsilon)re^\lambda a}{\theta^2},$$

$$\frac{\partial \lambda}{\partial \tau_0} + a \frac{\partial \lambda}{\partial r} = -a \left(\frac{e^\lambda - 1}{r} + \chi Pre^\lambda \right). \quad (6)$$

Eliminating ν from (5) with the aid of (6) and expressing σ in terms of P and V , we obtain

$$\frac{w}{\theta^2 c^2} (a_{\tau_0} + aa_r) + V \left(P_r + \frac{a}{c^2} P_{\tau_0} \right) = \frac{w}{2a} (a\lambda_r + \lambda_{\tau_0}),$$

$$-(V_{\tau_0} + aV_r) + \frac{V}{\theta^2} \left(a_2 + \frac{a}{c^2} a_{\tau_0} \right) + \frac{2aV}{r} = \frac{aV}{2} \left(\frac{a}{c^2} \lambda_{\tau_0} + \lambda_2 \right), \quad (7)$$

$$w_{\tau_0} + aw_r = V(P_{\tau_0} + aP_r), \quad \lambda_{\tau_0} = -\frac{\chi wre^\lambda a}{V\theta^2},$$

$$\nu_r = -\frac{1}{a}\lambda_{\tau_0} \left(1 + \frac{a^2}{c^2}\right) - \lambda_r, \quad w = \frac{k}{k-1}PV + ac^2.$$

In accordance with the general rules for finding self-similar solutions ^(4,5), we represent the sought functions in the form

$$a = \xi_1(z), \quad 1/V = t^{m_2}\xi_2(z), \quad P = t^{m_2}\xi_3(z),$$

$$e^\lambda = t^{m_3}\xi_4(z), \quad e^\nu = t^{m_4}\xi_5(z), \quad z = r/t. \quad (8)$$

It should be noted that the self-similar solution (8) is constructed using the independent variables r and t —the time of the central observer. We note that system (8) is less general than analogous equations in nonrelativistic gas dynamics.

The presence in equations (7) of the factors θ^2 and w forces one to choose the degree of t in the expression for a equal to zero, and the powers of t in the expressions for $1/V$ and P coincident.

Substituting (8) into (7), one can arrive at a system of 5 ordinary equations for the unknowns $\xi_1 - \xi_5$. In the process of obtaining the indicated system, no arbitrary powers of t remain ($m_2 = -2$, $m_3 = m_4 = 0$). The general self-similar solution containing 5 unknown quantities (a, V, P, λ, ν) will therefore contain 5 arbitrary constants entering the general solution of the system of 5 ordinary equations.

We note that, in order to determine the gas parameters, the first 4 equations of system (7) suffice. In this case the solution contains 4 arbitrary constants, i.e., the same number as is contained in the self-similar solution of gas motion in special relativity ⁽³⁾. The coincidence of the number of arbitrary constants is a consequence of the fact that the centrally symmetric motion of matter uniquely determines the metric of space.

It is easy to see that in the general case of the self-similar solution (8) there is no self-similar isentropic flow. Indeed, $\sigma = PV^k = \text{const}$ imposes an additional condition on the powers m_2 and m_3 , which contradicts the values of these same quantities found above.

Next we consider the ultrarelativistic flow of a gas. We shall now represent the sought functions in the form

$$a = \xi_1(z), \quad 1/V = t^{m_2}\xi_2(z), \quad P = t^{m_3}\xi_3(z),$$

$$e^\lambda = t^{m_4}\xi_4(z), \quad e^\nu = t^{m_5}\xi_5(z), \quad z = r/t. \quad (9)$$

When deriving the system of ordinary equations for the powers of t , the conditions

$$m_4 = m_5 = 0, \quad m_3 = -2 \quad (10)$$

are imposed.

Thus the exponent m_2 of V remains arbitrary, which makes it possible to satisfy the isentropy condition by setting $m_2 = -m_3/k$. The latter means that, in the case of an ultrarelativistic gas, for system (7) there exists a self-similar isentropic flow, in contrast to the general relativistic case, where such a flow does not exist.

The systems of ordinary equations obtained from (7) with the aid of (8) and (9) are, because of their cumbersomeness, inconvenient for investigation. It is therefore advisable to transform system (7) and then use the general investigation of self-similar solutions indicated above.

Let us pass in (7) to new independent variables λ, r . Then from (6) we have

$$e^{(\nu-\lambda)/2} a \frac{\partial t}{\partial \lambda} = -\frac{e^{-\lambda} r \theta^2}{\chi r^2 (P + \varepsilon)}. \quad (11)$$

With the aid of (11) we obtain from (7) the system

$$\begin{aligned} & \frac{1}{2\theta^2 c^2} \left[\frac{\partial a^2}{\partial \lambda} (e^{-\lambda} - 1 - \chi r^2 P) + r e^{-\lambda} \frac{\partial a^2}{\partial r} \right] - \frac{\omega^2}{c^2} \left[r \frac{\partial \ln V}{\partial r} e^{-\lambda} + \frac{\partial \ln V}{\partial \lambda} (e^{-\lambda} - 1 + \chi r^2 \varepsilon) \right] \\ &= \frac{1}{2} (e^{-\lambda} - 1 - \chi r^2 P) + \frac{T}{w} \frac{\partial \sigma}{\partial \lambda} \chi r^2 (P + \varepsilon), \\ & - \left[\frac{\partial \ln V}{\partial r} r e^{-\lambda} + \frac{\partial \ln V}{\partial \lambda} (e^{-\lambda} - 1 - \chi r^2 P) \right] + \frac{1}{2\theta^2} \left[r \frac{\partial \ln a^2}{\partial r} e^{-\lambda} + \frac{\partial \ln a^2}{\partial \lambda} (e^{-\lambda} - 1 + \chi r^2 \varepsilon) \right] + 2e^{-\lambda} \\ &= \frac{1}{2} [e^{-\lambda} - 1 + \chi r^2 \varepsilon], \quad (12) \end{aligned}$$

$$\frac{\partial \sigma}{\partial \lambda} (e^{-\lambda} - 1 - \chi r^2 P) + \frac{\partial \sigma}{\partial r} r e^{-\lambda} = 0,$$

$$\begin{aligned} & \frac{\partial \nu}{\partial r} r e^{-\lambda} + \frac{\partial \nu}{\partial \lambda} \left[\frac{\chi r^2}{\theta^2} (P + \varepsilon) + e^{-\lambda} - 1 - \chi r^2 P \right] \\ &= \frac{a^2}{c^2} \chi r^2 \frac{P + \varepsilon}{\theta^2} - (e^{-\lambda} - 1 - \chi r^2 P). \end{aligned}$$

We shall further consider an isentropic flow of an ultrarelativistic gas, when

$$P = (k-1)\varepsilon, \quad P = \sigma V^{-k}, \quad \sigma = \text{const}, \quad \varepsilon(k-1) = \sigma V^{-k}. \quad (13)$$

As was noted above, λ, a, ν are functions of z ; hence, in order to obtain a self-similar solution of system (12), one must set

$$a = a(\lambda), \quad \nu = \nu(\lambda), \quad P = r^{-2}A_1(\lambda). \quad (14)$$

Substituting (13), (14) into (12), we obtain

$$\begin{aligned} & \frac{1}{2\theta^2 c^2} \frac{da^2}{d\lambda} (e^{-\lambda} - 1 - \chi A_1) - \frac{\omega^2}{c^2} \left[\frac{2}{k} e^{-\lambda} - \frac{A_1^{-1}}{k} \frac{dA_1}{d\lambda} \left(e^{-\lambda} - 1 + \frac{\chi A_1}{k-1} \right) \right] = \frac{1}{2} (e^{-\lambda} - 1 - \chi A_1), \\ & - \left[\frac{2}{k} e^{-\lambda} - \frac{A_1^{-1}}{k} \frac{dA_1}{d\lambda} (e^{-\lambda} - 1 - \chi A_1) \right] + \frac{1}{2\theta^2} \frac{d \ln a^2}{d\lambda} \left(e^{-\lambda} - 1 + \frac{\chi A_1}{k-1} \right) + 2e^{-\lambda} \\ & = \frac{1}{2} \left(e^{-\lambda} - 1 + \frac{\chi A_1}{k-1} \right), \quad (15) \\ & \frac{d\nu}{d\lambda} \left(\frac{\chi}{\theta^2} A_1 \frac{k}{k-1} + e^{-\lambda} - 1 - \chi A_1 \right) = \frac{a^2}{\theta^2 c^2} \chi \frac{k A_1}{k-1} - (e^{-\lambda} - 1 - \chi A_1). \end{aligned}$$

Let us consider, in conclusion, the self-similar motion of dust matter in its own gravitational field. In its own reference frame such a problem has the exact solution ⁽¹⁾; however, it is of interest to obtain the same solution in the reference frame of a central observer, who, in addition to the distribution of the matter density, also observes the distribution of velocities.

Putting $P = 0$ in (12) and taking into account $a = a(\lambda)$, from the first equation of the indicated system we have

$$a^2/c^2 = 1 - e^{-\lambda} \quad (16)$$

(the integration constant is chosen so that at infinity, as $\lambda \rightarrow 0$, $u \rightarrow 0$).

Taking account of (16) and the relation following from (8),

$$1/V = t^{-2}B(z) = r^{-2}F(\lambda);$$

from the second equation of system (12) we have

$$\rho = 1/V = 1/r^2 [c_1 - \chi c^2 F(z)], \quad (17)$$

$$z = \frac{e^{-\lambda}}{2}, \quad F(z) = \int \frac{e^{-z} dz}{(1-2z)}. \quad (18)$$

It follows from (18) and (17) that as $r \rightarrow \infty$ ($e^\lambda \rightarrow 1$), $\rho \rightarrow 0$. Using (16) and (17), we determine $v(\lambda)$ from the last equation of system (12), after which from (11) we find $t(\lambda, r)$.

Let us note in conclusion that certain questions concerning self-similar motions of a gas in general relativity were considered in the dissertation of V. A. Skripkin⁽⁶⁾ in connection with the study of shock waves.

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Note: Figure translations are in progress. See original paper for figures.

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