

ON THE CARDINALITY OF HAUSDORFF SPACES

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Abstract

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MATHEMATICS

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ON THE CARDINALITY OF HAUSDORFF SPACES

(Presented by Academician P. S. Aleksandrov on 26 II 1965)

In this note it is proved that, with the increase of the cardinality of a Hausdorff space X , the upper bound of the cardinalities of its discrete, separated, and dispersed subspaces increases. The order of growth is established for each of these three types. For discrete spaces the order of growth is not less than $\log \log \log |X|$, for separated spaces $\log \log |X|$, and for dispersed spaces $\log |X|$, where $|X|$ denotes the cardinality of the set X , and $\log n$ is the least cardinal number m for which $\exp m = 2^m \geq n$.^{*} Hence it follows, in particular, that the cardinality of every Hausdorff space satisfying Suslin's condition^{**} hereditarily does not exceed $\exp \exp \exp \aleph_0$. We note that there exist bicompsacts of arbitrarily high cardinality satisfying Suslin's condition, for example dyadic ones⁽²⁾. On the other hand, for dyadic bicompsacts the hereditary fulfillment of Suslin's condition is equivalent to their metrizability⁽³⁾.

Let m be an infinite cardinal number, and j a natural number. Put $\varphi_j(m) = \exp \dots \exp \aleph_\rho$, if m is a number of the form $\aleph_{\rho+1}$, or

$$\varphi_j(m) = \sum_{\alpha \in A} \underbrace{\exp \dots \exp}_j m_\alpha,$$

if

$$m = \lim_{\alpha \in A} m_\alpha$$

and $m_\alpha < m$. Further, put

$$\psi_j(m) = \underbrace{\exp \dots \exp}_j m \quad \text{and} \quad \lg_j(m) = \underbrace{\log \dots \log}_j m.$$

Lemma 1. Let m_1 be an infinite cardinal number. Put $\varphi_2(m_1) = m_2$, and let m_3 be the least cardinal number greater than m_2 . Let

$$F_1 \supset F_2 \supset \dots \supset F_\alpha \supset \dots, \quad \alpha < \omega(m_3),$$

be a strictly decreasing well-ordered sequence of closed sets of a Hausdorff space X . Then X contains a discrete subspace T of cardinality m_1 .

Proof. Put $\Phi_\alpha = F_\alpha \setminus F_{\alpha+1}$. We note that all Φ_α , $1 \leq \alpha \leq \omega(m_3)$, are nonempty and pairwise disjoint sets. The construction of a discrete subspace $T \subset X$ of cardinality m_1 will be carried out by transfinite induction.

Take some point $x_1 \in \Phi_1$ and put $Ox_1 = X \setminus F_2$. Suppose that, for some ordinal $\xi < \omega(m_1)$, points

$$x_1, x_2, \dots, x_\beta, \dots, \quad \beta < \xi,$$

have been constructed such that $x_\beta \in \Phi_{\alpha(\beta)}$ and

$$\alpha(1) < \alpha(2) < \dots < \alpha(\beta) < \dots,$$

and, moreover, for every point x_β , $\beta < \xi$, a neighborhood Ox_β has been found possessing the following two properties: a) $Ox_\beta \not\ni x_\gamma$, if $\gamma < \beta$; b) $Ox_\beta \cap F_{\alpha(\beta)+1} = \emptyset$. Consider the space $\Phi = [R]_X$, if $R =$

* We note that the basic identity of logarithms in the domain of real numbers turns into an inequality in the domain of cardinal numbers, i.e. $\exp \log \geq$. For example, if $n = \aleph_0$, then $\log \aleph_0 = \aleph_0$ and $\exp \aleph_0 > \aleph_0$.

** One says that a space X satisfies Suslin's condition ⁽¹⁾ if every system of disjoint (i.e. pairwise nonintersecting) open sets in X is at most countable.

$$= \bigcup_{\beta < \zeta} x_\beta.$$

We note that R is everywhere dense in Φ , and

$$|R| = \left| \bigcup_{\beta < \zeta} x_\beta \right| = |\zeta| < \mathfrak{m}_1^*.$$

Hence, by Pospíšil's theorem ⁽⁴⁾, we obtain

$$|\Phi| \leq \exp \exp |\zeta| \leq \varphi_2(\mathfrak{m}_1) = \mathfrak{m}_2.$$

Since all the Φ_α are disjoint, while the cardinality of the family $\{\Phi_\alpha\}$, $\alpha < \omega(\mathfrak{m}_3)$, is equal to $\mathfrak{m}_3$, is regular, and $\mathfrak{m}_3 > \mathfrak{m}_2$, there exists a least ordinal number $\theta < \omega(\mathfrak{m}_3)$ such that

$$\Phi_\alpha \cap \Phi = \emptyset$$

for all $\alpha \geq \theta$. In this case put $\theta = \alpha(\zeta)$ and choose some point

$$x_\zeta \in \Phi_{\alpha(\zeta)}.$$

Let

$$Ox_\zeta = (X \setminus \Phi) \cap (X \setminus F_{\alpha(\zeta)+1}).$$

We note that, if $\beta < \zeta$, then all points x_β lie in Φ , and therefore $Ox_\zeta \not\ni x_\beta$, $\beta < \zeta$, i.e. a) is satisfied. On the other hand, by definition,

$$Ox_\zeta \cap F_{\alpha(\zeta)+1} = \emptyset,$$

i.e. b) is satisfied. Moreover, for every $\beta < \zeta$ we have

$$\alpha(\beta) < \alpha(\zeta),$$

since $\Phi_\alpha \cap \Phi = \emptyset$ for all $\alpha \geq \theta = \alpha(\zeta)$. Thus, for every $\beta < \omega(\mathfrak{m}_1)$ we can construct points x_β so that conditions a) and b) are fulfilled. The induction is complete.

Let

$$T = \{x_\beta\}, \quad 1 \leq \beta < \omega(\mathfrak{m}_1).$$

Since $x_\beta \in \Phi_{\alpha(\beta)}$ and all the Φ_α are disjoint, we have $|T| = \mathfrak{m}_1$. We show that T is a discrete subspace of X . Indeed, for an arbitrary point x_β consider the neighborhood Ox_β constructed above. By a), this neighborhood contains no point x_γ if $\gamma < \beta$. If, however, $\gamma > \beta$, then $\alpha(\gamma) > \alpha(\beta)$, and therefore $\alpha(\gamma) \geq \alpha(\beta) + 1$; hence

$$x_\gamma \in \Phi_{\alpha(\gamma)} \subset F_{\alpha(\gamma)} \subset F_{\alpha(\beta)+1}.$$

Therefore, by b), $Ox_\beta \not\ni x_\gamma$. Thus

$$T \cap Ox_\beta = x_\beta.$$

This means that T is discrete. The lemma is proved.

As is known, a topological space is called scattered if it contains no dense-in-itself (i.e. perfect) subset.

Lemma 2.** *Let $\mathfrak{m}$ be an infinite cardinal number and let*

$$F_1 \supset F_2 \supset \dots \supset F_\alpha \supset \dots, \quad \alpha < \omega(\mathfrak{m}),$$

be a strictly decreasing well-ordered sequence of closed sets of a Hausdorff space X . Then X contains a scattered subspace R of cardinality $\mathfrak{m}$.

Proof. We note that, for each α ,

$$F_\alpha \setminus F_{\alpha+1} = \Phi_\alpha$$

are nonempty pairwise disjoint sets. Choose one point

$$x_\alpha \in \Phi_\alpha.$$

Denote the resulting set by R . It is asserted that the set R is a scattered subspace of X . Indeed, suppose that R contains a dense-in-itself subset $M \subset R$. Let

$$x_\nu \in M$$

be the point with the least index among those belonging to M . Then, on the one hand,

$$x_\nu \in [M \setminus x_\nu]_X,$$

and on the other hand

$$x_\nu \in \Phi_\nu = F_\nu \setminus F_{\nu+1}.$$

Since ν is the least index, $M \setminus x_\nu \subset F_{\nu+1}$, i.e.

$$[M \setminus x_\nu]_X \subset F_{\nu+1},$$

which is impossible, since $x_\nu \notin F_{\nu+1}$. The lemma is proved.

Lemma 3. *Let $\mathfrak{m}$ be an infinite cardinal number. Suppose that every strictly decreasing well-ordered sequence*

$$F_1 \supset F_2 \supset \dots \supset F_\alpha \supset \dots, \quad \alpha < \omega(\mathfrak{m}),$$

of closed sets of a Hausdorff space X is stationary beginning with some $\alpha_0 < \omega(\mathfrak{m})$. Then

$$|X| \leq \varphi_1(\mathfrak{m}).$$

This lemma, for the case $\mathfrak{m} = \aleph_1$, was first proved by P. S. Alexandrov and P. S. Urysohn ⁽⁶⁾. For the case $\mathfrak{m} = \aleph_{\rho+1}$, it was proved by Yu. M. Smirnov ⁽⁷⁾. In the case where

$$\mathfrak{m} = \lim_{\alpha} \mathfrak{m}_\alpha$$

and $\mathfrak{m}_\alpha < \mathfrak{m}$, we note that, since for an arbitrary cardinal number $\mathfrak{m}_\alpha < \mathfrak{m}$ the cardinality of all possible sequences of 0' s and 1' s of length $\mathfrak{m}_\alpha$ is equal to $\exp \mathfrak{m}_\alpha$, the cardinality of all sequences of 0' s and 1' s of arbitrary length less than $\mathfrak{m}$ does not exceed

$$\sum_{\alpha} \exp \mathfrak{m}_\alpha = \varphi_1(\mathfrak{m}).$$

Further, since each point $x \in X$ is completely determined by some such sequence (see the proof of Theorem 6 in ⁽⁷⁾, p. 177), it follows that

$$|X| \leq \varphi_1(\mathfrak{m}).$$

As is known, a cardinal number $\mathfrak{m} > 1$ is called a caliber (N. A. Shanin) of a space X if every family of cardinality $\mathfrak{m}$, consisting of

* $|\xi|$ denotes the cardinality of all ordinal numbers less than ξ .

** This lemma for Fréchet-Urysohn spaces was first proved by V. Sierpiński ⁽⁵⁾.

consisting of nonempty open subsets of X , has an equipotent subfamily with nonempty intersection. We shall call a topological space X separated if $|X|$ is not a caliber of X . Let us note that every discrete space is scattered and separated. However, there exist scattered spaces that are not separated, and, conversely, there exist separated spaces that are not scattered. We record without proof the following fact. There exists a separated Hausdorff space X of cardinality $\aleph_1$ satisfying Suslin's condition.

Let X be a Hausdorff space. Denote by $\mathfrak{B}_i$, respectively, the set of all discrete ($i = 1$), separated ($i = 2$), or scattered ($i = 3$) subspaces of X . To each $Y_\alpha \in \mathfrak{B}_i$ assign the cardinal number $|Y_\alpha|$. Since the cardinality of a space is a weight characteristic in the sense of paper ⁽⁸⁾, this correspondence, just defined, is a weight function $l_i(Y_\alpha)$ defined on the set $\mathfrak{B}_i$. Denote by $\mathfrak{a}_i = \sup l_i(Y_\alpha) = \sup |Y_\alpha|$, where $Y_\alpha \in \mathfrak{B}_i$.

Theorem. For every Hausdorff space X we have $|X| \leq \psi_{4-i}(\mathfrak{a}_i)$, $i = 1, 2, 3$.

Proof. We consider successively three cases.

First case ($i = 1$). Put $\mathfrak{a}_1 = \aleph_0$, $\mathfrak{m}_1 = \aleph_{\rho+1}$, $\mathfrak{m}_2 = \varphi_2(\mathfrak{m}_1)$, and let $\mathfrak{m}_3$ be the least cardinal number greater than $\mathfrak{m}_2$. Note that every strictly decreasing well-ordered sequence

$$F_1 \supset F_2 \supset \dots \supset F_\alpha \supset \dots, \quad \alpha < \omega(\mathfrak{m}_3),$$

consisting of closed subsets of X , is stationary beginning with some $\alpha_0 < \omega(\mathfrak{m}_3)$, for otherwise, by Lemma 1, X would contain a discrete subspace T of cardinality $\mathfrak{m}_1$, which is impossible, since $\mathfrak{m}_1 > \mathfrak{a}_1$. Hence, by Lemma 3, we have

$$|X| \leq \varphi_1(\mathfrak{m}_3) = \exp \mathfrak{m}_2 = \exp \varphi_2(\mathfrak{m}_1) = \exp \exp \exp \mathfrak{a}_1 = \psi_3(\mathfrak{a}_1).$$

Second case ($i = 2$). In this case we shall prove that the density sX^* does not exceed $\mathfrak{a}_2$, whence, by Pospíšil's theorem ⁽⁴⁾, it will follow that $|X| \leq \exp \exp \mathfrak{a}_2 = \psi_2(\mathfrak{a}_2)$. Suppose that $sX > \mathfrak{a}_2$; then $sX \geq \mathfrak{n}$, where $\mathfrak{n}$ is the least cardinal number greater than $\mathfrak{a}_2$. We construct a separated space $Y \subset X$, $|Y| = \mathfrak{n}$. Let

$$X = \{x_1, x_2, \dots, x_\mu, \dots, \mu < \omega(|X|)\}$$

be well ordered. Put $y_1 = x_1$. Suppose that, for some $\alpha < \omega(\mathfrak{n})$, points $y_1, y_2, \dots, y_\beta, \dots, \beta < \alpha$, belonging to Y , have been constructed in such a way that, for every $\beta < \alpha$, we have

$$y_\beta \notin \left[\bigcup_{\gamma < \beta} y_\gamma \right]_X.$$

Consider

$$F = \left[\bigcup_{\beta < \alpha} y_\beta \right]_X.$$

Since $|\alpha| \leq \mathfrak{a}_2$, while $sX \geq \mathfrak{n} > \mathfrak{a}_2$, we have $X \setminus F \neq \emptyset$. Choose the least index $\mu < \omega(|X|)$ such that $x_\mu \in X \setminus F$, and put $x_\mu = y_\alpha$. The induction is complete. Let Y be the set of all chosen y 's. Then, by construction, $|Y| = \mathfrak{n}$. Denote

$$U_\alpha = Y \setminus \left[\bigcup_{\beta < \alpha} y_\beta \right]_X.$$

Note that all $U_\alpha \neq \emptyset$ and that

$$U_1 \supset U_2 \supset \dots \supset U_\alpha \supset \dots, \quad \alpha < \omega(\mathfrak{n}),$$

is a strictly decreasing well-ordered sequence of open sets with empty intersection, i.e. $|Y|$ is not a caliber of Y . This means that Y is a separated space of cardinality $\mathfrak{n}$, which is impossible, since $\mathfrak{n} > \mathfrak{a}_2$. Hence $sX \leq \mathfrak{a}_2$, and, as was said above, this means that $|X| \leq \psi_2(\mathfrak{a}_2)$.

Third case ($i = 3$). Let $\mathfrak{m}$ be the least cardinal number greater than $\mathfrak{a}_3$. Then every strictly decreasing well-ordered sequence of closed subsets of X is stationary beginning with some $\alpha_0 < \omega(\mathfrak{m})$, for otherwise, by Lemma 2, there would exist in X a scattered subspace of cardinality $\mathfrak{m}$, which is impossible. Applying Lemma 3, we obtain

$$|X| \leq \varphi_2(\mathfrak{m}) = \exp \mathfrak{a}_3.$$

The theorem is completely proved.

* The density sX of a space X is the minimum of the cardinalities of everywhere dense subsets of X .

This theorem had already been proved when the very recent note of Isbell ⁽⁹⁾* appeared, in which, for a completely regular X and $i = 1$, an analogous result was obtained. We note that Isbell's method is not suitable for Hausdorff spaces.

Corollary 1. For a Hausdorff space X we have the inequality

$$\mathfrak{a}_i = \sup_{Y_\alpha \in \mathfrak{B}_i} l_i(Y_\alpha) \geq \lg_{4-i} |X|, \quad i = 1, 2, 3.$$

Indeed, suppose, for example, that $\mathfrak{a}_3 < \log |X|$; then, by the definition of the logarithm, $\exp \mathfrak{a}_3 < |X|$, which contradicts the theorem ($i = 3$). The cases $i = 2$ and $i = 1$ are considered analogously.

Corollary 2. If a Hausdorff space X satisfies Suslin's condition hereditarily, then

$$|X| \leq \psi_3(\aleph_0).$$

I do not know whether this estimate for the cardinality of X is sharp.

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* The second part of the assertion of Theorem 1 in note (⁹) is erroneous, since there exist (¹⁰), for every number $m \geq \aleph$, completely regular spaces of cardinality m and weight $\exp m$, which therefore cannot be embedded in I^m .

Note: Figure translations are in progress. See original paper for figures.

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