

# A priori $L^p$ estimates for solutions of parabolic equations

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**Abstract**

**Full Text**

**M. D. RAMAZANOV**

# **A PRIORI ESTIMATES OF $L^p$ -TYPE FOR SOLUTIONS OF PARABOLIC EQUATIONS**

*(Presented by Academician I. G. Petrovskii on 19 XI 1964)*

## **MATHEMATICS**

The paper studies a class of function spaces close in their properties to the spaces  $L^p$  and denoted below by  $V_{p,x}$ , which have proved convenient in describing general boundary-value problems for parabolic equations.

§ 1. Let  $f(x) = f(x_1, \dots, x_n)$  be a finite sufficiently smooth function in  $E^n$ , and let  $\tilde{f}(\vec{\alpha}) = \tilde{f}(\alpha_1, \dots, \alpha_n)$  be its Fourier transform. The Euclidean space formed by the set  $(\alpha_1, \dots, \alpha_n)$  of real arguments of the function  $\tilde{f}(\vec{\alpha})$  will be denoted by  $R^n$ . We divide the first octant of the space  $R^n$ —the set  $\Pi_{j=1}^n (0 < \alpha_j < \infty)$ —by the hyperplanes  $\alpha_j = 2^{m_j}$ ,  $m_j = 0, \pm 1, \pm 2, \dots$  ( $j = 1, \dots, n$ ), into parallelepipeds  $\Pi(\mathbf{m}) = \Pi(m_1, \dots, m_n)$ , and associate with each parallelepiped  $\Pi(\mathbf{m})$  the function  $f_m^{(1)}(x)$ , whose Fourier transform  $\tilde{f}_m^{(1)}(\vec{\alpha})$  coincides on  $\Pi(\mathbf{m})$  with the function  $\tilde{f}(\vec{\alpha})$  and is equal to zero in  $R^n \setminus \Pi(\mathbf{m})$ . Carrying out similar constructions for each  $j$ -th octant ( $j = 1, \dots, 2^n$ ), we associate with the function  $f(x)$  the vector-valued function with a countable number of components  $F(x) = \{f_m^{(j)}(x)\}$ ; here  $j = 1, \dots, 2^n$  is the number of the octant,  $\mathbf{m} = (m_1, \dots, m_n)$ ,  $m_k = 0, \pm 1, \pm 2, \dots$ .

It can be shown that, for a finite sufficiently smooth function  $f(x)$ , the quantity

$$\|f\| = \sum_{m,j} \|f_m^{(j)}(x)\|_{L^p(x)} = \|F(x)\|_{L^p(x)} \|1\|_{l^1} \quad (1 < p < \infty) \quad (1)$$

is finite.

The closure of finite sufficiently smooth functions in the norm (1) forms a Banach space, which we denote by  $V_{p,x}(E^n)$ , and the norm (1) by  $(f | V_{p,x}(E^n))$ . Since  $f(x) = \sum_{m,j} f_m^{(j)}(x)$ , we have

$$\|f\|_{L^p} = \left\| \sum_{m,j} f_m^{(j)}(x) \right\|_{L^p} \leq \sum_{m,j} \|f_m^{(j)}(x)\|_{L^p} = (f | V_{p,x}(E^n)),$$

i.e., the embedding  $V_{p,x}(E^n) \subset L^p$  holds. It can be shown that  $V_{p,x}(E^n) \neq L^p$ .

The relations between the Sobolev spaces  $W_{p,x}^k$  <sup>(1,2)</sup> and the spaces  $V_{p,x}(E^n)$  are established by the following theorem.

**Theorem 1.** Let  $f(x)$  be a finite function with support of diameter  $a$ . Then for any  $\varepsilon > 0$  and  $1 < p < \infty$  the inequality

$$(f | V_{p,x}(E^n)) \leq K(p, \varepsilon) [\ln(a+2)]^n \|f\|_{W_{p,x}^\varepsilon}$$

holds with a constant  $K(p, \varepsilon)$  independent of the function  $f(x)$ .

To characterize the differential properties of functions, we now define spaces with derivatives by means of Bessel potentials <sup>(3)</sup>, starting from the space  $V_{p,x}(E^n)$ . Namely, we shall say that a function  $f(x)$  belongs to the space

$$V_{p,x}^k(E^n) = V_{p,x_1,\dots,x_n}^{k_1,\dots,k_n}(E^n), \quad k_j \geq 0,$$

if there belongs to the space  $V_{p,\mathbf{x}}(E^n)$  a function  $g(\mathbf{x})$  whose Fourier transform is defined by the equality

$$\tilde{f}(\vec{\alpha}) \sum_{j=1}^n (|\alpha_j|^{k_j} + 1) = \tilde{g}(\vec{\alpha}).$$

As the norm of the function  $f(\mathbf{x})$  in the space  $V_{p,\mathbf{x}}^{\mathbf{k}}(E^n)$  we take the quantity

$$(f | V_{p,\mathbf{x}}^{\mathbf{k}}(E^n)) = (g | V_{p,\mathbf{x}}(E^n)).$$

For the spaces  $V_{p,\mathbf{x}}^{\mathbf{k}}(E^n)$  there hold exact theorems on the trace of a function on the subspace  $E^{n-1}$  and on the extension of a function from the subspace  $E^{n-1}$  to all of  $E^n$ .

**Theorem 2.** If the function  $f(\mathbf{x}) \in V_{p,\mathbf{x}}^{\mathbf{m}}(E^n)$  and  $m_n \geq 1/p$ , then the function  $f(\mathbf{x}', 0) \in V_{p,\mathbf{x}'}^{\mathbf{k}'}(E^{n-1})$ , where  $\mathbf{x}' = (x_1, \dots, x_{n-1})$ ,  $\mathbf{k}' = (k_1, \dots, k_{n-1})$ ,  $k_j = m_j(1 - 1/pm_n)$ ,  $j = 1, \dots, n-1$ .

**Theorem 3.** Suppose that on the hyperplane  $x_n = 0$  a function  $g(\mathbf{x}') \in V_{p,\mathbf{x}'}^{\mathbf{k}'}(E^{n-1})$  is given. Then for every  $l > 0$  there exists a function  $f_l(\mathbf{x})$ , which is defined in all of  $E^n$ , coincides on the hyperplane  $x_n = 0$  with the function  $g(\mathbf{x}')$  in the sense of the space  $V_{p,\mathbf{x}'}^{\mathbf{k}'}(E^{n-1})$ ,

$$f_l(\mathbf{x}) \in V_{p,\mathbf{x}}^{\mathbf{m}'}(E^n), \quad \text{where } m_j = k_j \left(1 + \frac{1}{pl}\right), \quad j = 1, \dots, n-1, \quad m_n = l + \frac{1}{p}.$$

Theorems of this type have been proved in the spaces  $W_{2,\mathbf{x}}^{\mathbf{k}}(E^n)$  <sup>(2)</sup>,  $H_{p,\mathbf{x}}^{\mathbf{k}}(E^n)$  <sup>(4)</sup>,  $B_{p,\theta,\mathbf{x}}^{\mathbf{k}}(E^n)$  <sup>(5)</sup>. At the same time it is known <sup>(5)</sup> that in the spaces  $W_{p,\mathbf{x}}^{\mathbf{k}}(E^n)$

for  $p \neq 2$  exact theorems on traces and extensions of functions cannot be established.

We shall call a function  $\Phi(\vec{\alpha})$  a **multiplier from the space  $V_{p,\mathbf{x}}(E^n)$  to the space  $V_{q,\mathbf{x}}(E^n)$**  if the operator  $Af = F^{-1}\Phi Ff$ , acting from  $V_{p,\mathbf{x}}(E^n)$  into  $V_{q,\mathbf{x}}(E^n)$ , is bounded. Here  $F$  is the Fourier-transform operator, and  $\Phi$  is the operator of multiplication by the function  $\Phi(\vec{\alpha})$ .

**Theorem 4.** Suppose that the function  $\Phi(\vec{\alpha})$  is continuous together with all derivatives of the form  $\partial^k \Phi / \partial \alpha_{j_1} \dots \partial \alpha_{j_k}$ , where  $j_l \neq j_m$  for  $l \neq m$ ;  $k = 1, \dots, n$ , outside the coordinate planes  $\alpha_s = 0$  ( $s = 1, \dots, n$ ), and suppose that the inequalities

$$\left| \alpha_1^{k_1+\beta} \dots \alpha_n^{k_n+\beta} \frac{\partial^k \Phi}{\partial \alpha_1^{k_1} \dots \partial \alpha_n^{k_n}} \right| \leq C,$$

are satisfied, where

$$\beta = \frac{1}{p} - \frac{1}{q} \geq 0;$$

$k_s = 0$  or  $1$ ,  $k_1 + \dots + k_n = k$  takes the values  $0, 1, \dots, n$ . Then the function  $\Phi(\vec{\alpha})$  is a multiplier from the space  $V_{p,\mathbf{x}}(E^n)$  to the space  $V_{q,\mathbf{x}}(E^n)$ .

The corresponding result for multipliers from  $L^p$  to  $L^q$  was established in <sup>(6)</sup>.

We shall also need the following

**Theorem 5.** Suppose that the function  $f(\mathbf{x}) \in V_{p,\mathbf{x}}(E^n)$ . Then the function  $c(\mathbf{x}) \cdot f(\mathbf{x}) \in V_{p,\mathbf{x}}(E^n)$ , if  $c(\mathbf{x})$  is finite and satisfies the conditions

$$\begin{aligned} & \max_{x_{j_{k+1}}, \dots, x_{j_n}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} |\tilde{c}(\alpha_{j_1}, \dots, \alpha_{j_k}; x_{j_{k+1}}, \dots, x_{j_n})| \times \\ & \times \prod_{l=1}^k \ln(|\alpha_{j_l}| + 2) d\alpha_{j_1} \dots d\alpha_{j_k} \leq C \end{aligned}$$

for all possible sets  $(j_1 \dots j_k)$ ,  $k = 0, \dots, n$ . Here  $c(\alpha_{j_1}, \dots, \alpha_{j_k}; x_{j_{k+1}}, \dots, x_{j_n})$  denotes the Fourier transform of the function  $c(x_1, \dots, x_n)$  with respect to the variables  $x_{j_1}, \dots, x_{j_k}$ .

§ 2. Consider the space  $E^{n+1}$ ; a point of this space will be denoted by  $(t, \mathbf{x})$ . Let  $Q$  be a compact domain in  $E^{n+1}$ , contained between the planes  $t = 0$  and  $t = T$  and bounded by a smooth surface  $S$ . We assume that for each  $t \in [0, T]$  the set  $\omega_t = Q \cap (t = \text{const})$  is a simply connected domain with smooth boundary  $S \cap (t = \text{const})$ , and that the surface  $S$  is such that at every point at which the normal to  $S$  is parallel to the axis  $Ot$ , the tangent plane has contact with the

surface  $S$  of order no higher than some number  $\theta$ , fixed for the given domain  $Q$ . We shall call  $\theta$  the order of singularity of the surface  $S$ .

In the domain  $Q$  consider the parabolic equation

$$Lu \equiv u_t + \sum_{|k| \leq 2m} a_k(t, \mathbf{x}) D_x^k u = f(t, \mathbf{x}),$$

$$\operatorname{Re} \sum_{|k|=2m} a_k(t, \mathbf{x}) (i\vec{\alpha})^k \geq \delta \sum_{j=1}^n \alpha_j^{2m} \quad (2)$$

for all real  $\alpha_j$ ,  $j = 1, \dots, n$ , with a constant  $\delta > 0$ , the same for all  $(t, \mathbf{x}) \in Q$ . Here  $k = (k_1, \dots, k_n)$ ,  $k_j$  are integers,  $k_j \geq 0$ ,  $|k| = k_1 + \dots + k_n$ ,  $D_x^k u = D_{x_1}^{k_1} \dots D_{x_n}^{k_n} u$ ,  $(i\vec{\alpha})^k = (i\alpha_1)^{k_1} \dots (i\alpha_n)^{k_n}$ .

Let  $u(t, \mathbf{x})$  be, in  $Q$ , a solution of (2) satisfying the boundary conditions

$$u(0, \mathbf{x}) = \psi(\mathbf{x}) \quad \text{for } \mathbf{x} \in \omega_0; \quad (3)$$

$$B_j(t, \mathbf{x}, D_t, D_x)u|_S = \varphi_j(t, \mathbf{x})|_S \quad (j = 1, \dots, m),$$

where  $B_j(t, \mathbf{x}, D_t, D_x)$  is a linear differential operator with variable coefficients of order  $\nu_j$  (with derivatives in  $t$  counted with weight  $2m$ ). We shall assume that the operators  $B_j$  satisfy the condition of linear independence modulo the principal part of the operator  $L$ . A precise formulation of this condition can be found, for example, in [7].

The coefficients of the equation and of the boundary operators are assumed to be sufficiently smooth functions.

For a function  $f$  given in the domain  $Q$ , a function  $\varphi$  given on the surface  $S$ , and a function  $\psi$  given on  $\omega_0$ , introduce the following norms:

$$(f | V_{p,t,\mathbf{x}}^{k_0, \mathbf{k}}(Q)) = \inf_g (g | V_{p,t,\mathbf{x}}^{k_0, \mathbf{k}}(E^{n+1})), \quad \text{where } g = f \text{ in } Q;$$

$$(\varphi | V_{p,t,s}^{k_0, k}(S)) = \inf_g (g | V_{p,t,\mathbf{x}}^{k_0+k_0/pk, k+1/p}(E^{n+1})), \quad \text{where } g = \varphi \text{ on } S;$$

$$(\psi | V_{p,\mathbf{x}}^k(\omega_0)) = \inf_g (g | V_{p,t,\mathbf{x}}^{k/2m+1/p, k+2m/p}(E^{n+1})), \quad \text{where } g = \psi \text{ on } \omega_0.$$

**A priori estimate theorem.** Let a sufficiently smooth function  $u(t, \mathbf{x})$  satisfy equation (2) and the boundary conditions (3). Then for any  $\varepsilon > 0$  and  $\gamma \geq \max(2m, \nu_j + 1/p)$  the estimate

$$(u \mid V_{p,t,\mathbf{x}}^{\gamma/2m,\gamma}(Q)) \leq K^{(\varepsilon)} \left\{ (f \mid V_{p,t,\mathbf{x}}^{\gamma+\varepsilon/2m-1, \gamma+\varepsilon-2m}(Q)) + \right. \\ \left. + (\psi \mid V_{p,\mathbf{x}}^{\gamma+\varepsilon-2m/p}(\omega_0)) + \sum_{j=1}^m \left( \varphi_j \mid V_{p,t,s}^{\frac{\gamma+\varepsilon-\nu_j-1/p}{2m}, \gamma+\varepsilon-\nu_j-1/p}(S) \right) \right\}, \quad (4)$$

where the constant  $K$  does not depend on the function  $u(t, \mathbf{x})$ , if  $\theta$ , the order of singularity of the surface  $S$ , satisfies the conditions:

$$\theta < 2m \quad \text{when} \quad \frac{\gamma}{2m} \leq \frac{1}{p} - \frac{1}{2m};$$

$$0 < \frac{1}{\gamma/2m - 1/p} \quad \text{when} \quad \frac{1}{p} + \frac{1}{2m} \leq \frac{\gamma}{2m} < 1 + \frac{1}{p}; \quad \theta = 1 \quad \text{when} \quad 1 + \frac{1}{p} \leq \frac{\gamma}{2m}.$$

For a given  $\gamma$  one can precisely calculate the smoothness of the coefficients of the equation and of the boundary operators, and the smoothness  $S$ , so that inequality (4) is satisfied.

The proof of the theorem is based on the properties of the spaces  $V_{p,x}^k$  established in Theorems 1-5.

Theorem 1 shows that from formula (4) one can obtain an a priori estimate in the norms  $W_{p,x}^k$  of Sobolev spaces with arbitrary  $\varepsilon$ -accuracy.

For the straight cylinder, a priori estimates of the solution were obtained in papers (<sup>8, 9, 10</sup>): in the norms  $W_{2,t,x}^{\gamma/2m,\gamma}$  of the spaces (<sup>8</sup>), in the norms  $W_{2,t,x}^{k/2m,k}$  for  $k$  a multiple of  $2m$  (<sup>9</sup>), and in the norms  $W_{p,t,x}^{\gamma/2m,\gamma}$  for any  $\gamma \geq 2$ , but for an equation of second order (<sup>10</sup>).

A parabolic equation in domains with boundaries having singular points was first considered in (<sup>11</sup>), where, for the heat-conduction equation, a classical solution of the first boundary-value problem was constructed.

For a general parabolic equation, the first boundary-value problem in domains having singularities on the boundary was studied in (<sup>12, 13</sup>).

Taking in formula (1) the outer norm of the function  $F(x)$  in the space  $l^q$  ( $1 \leq q \leq \infty$ ), one can arrive at new spaces  $l^q(L^q)$ . We have considered in detail the space  $l^1(L^p) \equiv V_p(E^n)$ ; however, similar results are also established in the spaces  $l_p(L^q)$  for any  $1 \leq q \leq \infty$ , and for parabolic equations a priori estimates analogous to those formulated above are obtained.

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