

On the Asymptotic Properties of the Spectral Function of Hypoelliptic Operators

V. N. Gorchakov

1965

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196501.59191>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

V. N. GORCHAKOV

ON THE ASYMPTOTIC PROPERTIES OF THE SPECTRAL FUNCTION OF HYPOELLIPTIC OPERATORS

(Presented by Academician A. Yu. Ishlinskii on 23 VII 1964)

MATHEMATICS

1°. In this note we study the asymptotic behavior of the spectral function of hypoelliptic operators in n variables satisfying condition (2) (see § 2°). This class is considerably broader than that considered in paper ⁽¹⁾. The sharper estimates of the fundamental solution used here make it possible to obtain results which, in particular, strengthen Theorem 2 of paper ⁽¹⁾.

2°. We shall denote a point of n -dimensional real Euclidean space R^n by $x = (x_1, \dots, x_n)$, $|x| = \sqrt{x_1^2 + \dots + x_n^2}$; a point of n -dimensional complex space C^n by $s = \sigma + i\tau$, where $\sigma, \tau \in R^n$, $|s| = \sqrt{|\sigma|^2 + |\tau|^2}$. Let D be a domain in R^n , $C_0^k(D)$ the set of finite functions in D having k continuous derivatives, $C_0^\infty(D)$ the set of finite functions infinitely differentiable in D , and $C_0(D)$ the finite continuous functions in D .

Consider on functions $u \in C_0^\infty(D)$ the hypoelliptic differential operator

$$P\left(\frac{1}{i}\frac{\partial}{\partial x_1}, \dots, \frac{1}{i}\frac{\partial}{\partial x_n}\right),$$

whose characteristic polynomial $P(s_1, \dots, s_n) = P(s)$ is complete (see ⁽²⁾) and has constant real coefficients. Denote by P_0 the minimal operator ⁽²⁾, and by $\hat{P}$ any semibounded self-adjoint extension of it. Let $\theta(x, y, \lambda)$ be the spectral function of the operator $\hat{P}$. If $D = R^n$, then the spectral function, as is known ⁽³⁾, is expressed by the formula

$$\theta_0(x - y, \lambda) = \frac{1}{(2\pi)^n} \int_{P(\sigma) < \lambda} \exp\left[-i \sum (x_k - y_k) \sigma_k\right] d\sigma_1 \dots d\sigma_n. \quad (1)$$

Finally, let λ_0 be a number less than the lower bounds of the operators $\hat{P}$ and P_0 , defined on all of R^n .

Introduce the class K of operators satisfying, for every $c > 1$ and all sufficiently large λ , the inequality

$$\frac{\theta_0(0, c\lambda)}{\theta_0(0, \lambda)} \leq c^\gamma \quad (2)$$

with some $\gamma > 0$ (depending on the operator). This class is sufficiently broad*. In particular, it includes: a) all elliptic and quasi-elliptic operators; b) operators satisfying, for sufficiently large r , the condition

$$\frac{P'_r(r, \varphi)}{P(r, \varphi)} \geq \frac{A}{r},$$

* The question of whether there exist hypoelliptic operators not belonging to K remains open.

where $(r, \varphi) = (r, \varphi_1, \dots, \varphi_{n-1})$ are polar coordinates in R^n , and $A > 0$ does not depend on φ . It is easy to see that condition b) is satisfied, for example, by hypoelliptic polynomials in n variables with positive coefficients and even degrees $\sigma_1, \dots, \sigma_n$, in particular by the three-term operators considered in (1).

It can be shown that the class K also contains operators satisfying neither of the conditions a), b).

Let us formulate the main results obtained in the present paper.

Theorem 1. *If a hypoelliptic operator belongs to the class K , then as $\lambda \rightarrow +\infty$*

$$\theta(x, x, \lambda) \sim \theta_0(0, \lambda)$$

uniformly on compact subsets of the domain D .

Theorem 2. *For any hypoelliptic operator, as $\lambda \rightarrow +\infty$, the asymptotic estimate holds for every real $k \geq 1$*

$$I^k \theta(x, y, \lambda) = I^k \theta_0(x - y, \lambda) + o(\lambda^{(\beta - k\nu)/p}),$$

uniformly on each compact subset of the set $D \times D$.

Here $I^k \sigma(\lambda)$ denotes the Riesz mean of order k of the function σ on the interval (λ_0, λ) , p is the degree of $P(s)$, and β and ν are positive constants depending on the operator ($\nu \leq 1$).

For the proof of these theorems we shall need a number of auxiliary assertions.

3°. Lemma. *For a given hypoelliptic polynomial $P(s)$ of degree p with real coefficients ($s_j = \sigma_j + i\tau_j$; $j = 1, \dots, n$), one can choose positive constants $a, b, h \leq p$ and $\nu \leq 1$ such that throughout the space C^n , for $t > 0$, the inequality*

$$|e^{-tP(s)}| \leq C \exp \left[-ta \sum_{j=1}^n |\sigma_j|^h + tb \sum_{j=1}^n |\tau_j|^{p/\nu} \right]. \quad (3)$$

Let $0 < \varepsilon < 1$. Using Taylor's formula for the polynomial, we can write

$$|\exp[-tP(s)]| \exp[t(1-\varepsilon)P(\sigma)] = \exp \left[-\varepsilon tP(\sigma) - \sum_q \frac{D^q P(\sigma)}{q!} \tau^q \right], \quad (4)$$

where $D^q = \partial^{q_1+\dots+q_n}/\partial\sigma_1^{q_1}\dots\partial\sigma_n^{q_n}$, $|q| = \sum q_j$ are even numbers; $\tau^q = \tau_1^{q_1}\dots\tau_n^{q_n}$; $q! = q_1!\dots q_n!$. Consider the region $S_{r_0}\{|\sigma| > r_0\}$, where r_0 is sufficiently large. Since for hypoelliptic operators with real coefficients, for large $|\sigma|$,

$$\min_{|\sigma|=r} P(\sigma) = a_1 r^h (1 + o(1))$$

with some $a_1 > 0$, $h > 0$ (h rational) (see (4)), the left-hand side of equality (4) in this region can be estimated from below by the function $C_1 |\exp[-tP(s)]| \exp[ta|\sigma|^h]$, $a > 0$. It can be shown that for each hypoelliptic operator there is such a $0 < \nu \leq 1$ that in the region $S_\nu\{|\tau| \leq c_1|\sigma|^\nu; |\sigma| > r_0\}$, for sufficiently large r_0 , the right-hand side of (4) will be bounded above by some constant, and in the region $S_{r_0} - S_\nu$ by the function $C_2 \exp[C_3 t|\tau|^{p/\nu}]$. Finally, if r_0 has already been chosen in the indicated way, then for $|\sigma| \leq r_0$, in view of the boundedness of $D^q P(\sigma)$, one easily obtains the estimate

$$|\exp[-tP(s)]| \leq C_4 \exp[-at|\sigma|^h + C_5 t|\tau|^p].$$

Combining all these estimates into one and taking into account the elementary inequality

$$0 < m \leq |\xi|^l / \sum |\xi_k|^l \leq M, \quad \xi \in R^n,$$

we obtain the assertion of the lemma.

4°. Consider the equation, parabolic in the sense of G. E. Shilov (5),

$$\frac{\partial u}{\partial t} = P \left(\frac{1}{i} \frac{\partial}{\partial x} \right) u,$$

where P is an arbitrary hypoelliptic operator with constant real coefficients. We derive estimates for the derivatives with respect to x and t of its fundamental solution

$$\begin{aligned} G_0(t, x_1, \dots, x_n) = \\ = \frac{1}{(2\pi)^n} \int_{R^n} \exp \left[-tP(\sigma_1, \dots, \sigma_n) - i \sum x_k \sigma_k \right] d\sigma_1 \dots d\sigma_n, \end{aligned} \quad (5)$$

using the device applied in (5) for functions of one variable. In view of inequality (3), by Cauchy's theorem, the value of the integral will not change if one integrates along the horizontal straight lines $\tau_k = \alpha_k$ ($k = 1, \dots, n$). In view of the separation of variables in estimate (3),

$$|D_x^q G_0(t, x)| \leq \prod_{k=1}^n \int_{-\infty}^{\infty} |s_k|^{q_k} \exp[-at|\sigma_k|^h + bt|\tau_k|^{p/\nu}] \exp[-x_k \tau_k] d\sigma_k.$$

For each of the factors we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(\sqrt{\sigma_k^2 + \tau_k^2} \right)^{q_k} \exp[-at|\sigma_k|^h + bt|\tau_k|^{p/\nu}] \exp[-x_k \tau_k] d\sigma_k \leq \\ & \leq B_k \exp[-x_k \tau_k + tb|\tau_k|^{p/\nu}] \int_{-\infty}^{\infty} (|\sigma_k|^{q_k} + |\tau_k|^{q_k}) \exp[-at|\sigma_k|^h] d\sigma_k. \end{aligned} \quad (6)$$

Choose the quantity τ_k so that

$$|\tau_k| = \left(|x_k| \frac{pb}{\nu} t \right)^{\nu/(p-\nu)},$$

and its sign so that $x_k \tau_k = |x_k| \cdot |\tau_k|$. Substituting τ_k in the right-hand side of (6) and multiplying the estimates, we obtain

$$|D_x^q G_0(t, x)| \leq \prod_{k=1}^n B_k \left(\frac{C'_k}{t^{(1+q_k)/h}} + \frac{|x_k|^{q_k \nu/(p-\nu)}}{t^{q_k \nu/(p-\nu)+1/h}} \right) \exp \left[-C'_k \frac{|x_k|^{p/(p-\nu)}}{t^{\nu/(p-\nu)}} \right]. \quad (7)$$

Analogous estimates are also obtained for derivatives with respect to t . Since for hypoelliptic operators $0 < \nu \leq 1$, $p \geq 2$, it follows that $\nu(p-\nu) < 0$.

5°. Consider in $C_0(D)$ the Hermitian bilinear form

$$V(t, f, g) = ((e^{-t\hat{P}} - e^{-tP_0})f, g).$$

Its kernel, in view of the formula

$$(E_\lambda f, g) = \int_{D \times D} \theta(x, y, \lambda) f(x) g(y) dx dy$$

is the difference

$$v(t, x, y) = G(t, x, y) - G_0(t, x - y), \quad (8)$$

where $G(t, x, y)$ is the Laplace-Stieltjes transform of the spectral function $\theta(x, y, \lambda)$ of the operator $\hat{P}$ (the existence of $G(t, x, y)$ follows from Theorem 3.9 of Hermander ⁽²⁾). Let Q be a fixed compact subset of the domain D . In view of the boundedness of the kernel $e^{-t\lambda}$, for any $t > 0$ and any $f, g \in C_0(Q)$,

$$|V(t, f, g)| \leq C_Q \sup_{x \in Q} |f(x)| \sup_{x \in Q} |g(x)|, \quad (9)$$

where C_Q does not depend on t . Let W be any compact subset of the domain D . Choose an open set $U \subset D$ with compact closure, containing W . Further, let $x \in U$ and let F be a compact subset of D containing $\bar{U}$. Take $h(x) \in C_0^\infty(D)$ with support F , $h(x) = 1$ on U , and construct the function

$$b(t, x, y) = \left[P \left(\frac{1}{i} \frac{\partial}{\partial y} + \frac{\partial}{\partial t} \right) \right] [G_0(t, x - y)(1 - h(y))]. \quad (10)$$

Obviously, $b(t, x, y)$ is a finite function in y with support contained in $Q = F \setminus U$ for every $x \in U$, and $b(t, x, y) \rightarrow 0$ as $t \downarrow 0$ for each $(x, y) \in U \times D$. We apply Gårding's identity ⁽³⁾, which in our notation (8) and (10) has the form

$$v(t, x, y) = \int_0^t dt' \int_0^{t'} dt'' \int_{D \times D} v(t'', \xi, \eta) b(t - t', x, \xi) b(t' - t'', x, \eta) d\xi d\eta \quad (11)$$

and is valid for any $(x, y) \in U \times U$.

By virtue of inequality (9) and identity (11), for $0 < t < t_0$,

$$|v(t, x, y)| \leq C_Q t^2 \sup_{0 \leq \tau \leq t} \sup_{\xi \in Q} |b(\tau, x, \xi)| \sup_{0 \leq \tau \leq t} \sup_{\eta \in Q} |b(\tau, y, \eta)|.$$

Using estimates (7), the last inequality and the fact that, for all points $x \in W$ and $\xi \in Q$, $0 < c' \leq |x - \xi| \leq C'$, we obtain, as $t \downarrow 0$, the asymptotic estimate

$$|G(t, x, y) - G_0(t, x - y)| = o \left(\exp \left[-\frac{C}{t^{\nu/(p-\nu)}} \right] \right), \quad (12)$$

uniformly in $(x, y) \in W \times W$, with some constant $C > 0$.

6°. We pass to the direct proof of the theorems formulated in Section 2°. Theorem 1 follows immediately from estimate (12) and B. I. Korenblyum's Tauberian theorem for ratios of functions ⁽⁶⁾.

For the proof of Theorem 2, Ganelius' Tauberian theorem is applied to the function $\sigma(\lambda) = \theta(x, y, \lambda) - \theta_0(x - y, \lambda)$, with remainder estimate (7). Put $\Lambda = (\lambda, \lambda + \lambda^{(p-\nu)/p})$. From the relations

$$\sup_{\mu \in \Lambda} \int_{\lambda}^{\mu} d\sigma(\omega) \leq \text{Var}_{\Lambda} \sigma(\lambda) \leq \text{Var}_{\Lambda} \theta(x, y, \lambda) + \text{Var}_{\Lambda} \theta_0(x - y, \lambda),$$

$$\text{Var}_{\Lambda} \theta(x, y, \lambda) \leq [\text{Var}_{\Lambda} \theta(x, x, \lambda) \text{Var}_{\Lambda} \theta(y, y, \lambda)]^{1/2}$$

it follows that the Tauberian condition of the theorem can be obtained by estimating the variations of the functions $\theta(x, x, \lambda)$ and $\theta_0(0, \lambda)$. Because these functions are nondecreasing, we obtain the rough estimate

$$\sup_{\mu \in \Lambda} \int_{\lambda}^{\mu} d\sigma(\omega) = o(\lambda^{\alpha}),$$

where α is some positive exponent depending on the operator. Now Ganelius' theorem gives

$$I^k \sigma(\lambda) = o \left(\lambda^{\alpha + (\nu - p)/p} (\lambda - \lambda_0) \frac{\lambda^{k(p - \nu)/p}}{(\lambda - \lambda_0)^k} \right) = o(\lambda^{\beta - k\nu/p}),$$

which was to be proved. Since $\nu/p > 0$, for sufficiently large k the Riesz mean of order k of the function $\sigma(\lambda)$ tends to zero uniformly in $(x, y) \in W \times W$.

In conclusion, the author expresses deep gratitude to B. I. Korenblyum and Yu. M. Berezansky for valuable advice and attention to the present work.

Kyiv Civil Engineering Institute

Received
20 VII 1964

REFERENCES

1. V. N. Gorchakov, DAN, **152**, No. 3 (1963).
2. L. Hörmander, *On the theory of general partial differential operators*, Moscow, 1959.
3. L. Gårding, Kungl. Fys. Sällsk. i Lund Förh., **24**, 21 (1954).
4. E. A. Gorin, UMN, **16**, 1 (1961).

5. I. M. Gel' fand, G. E. Shilov, *Some Questions in the Theory of Differential Equations*, Moscow, 1958.
6. B. I. Korenblyum, UMN, **8**, 3 (1953).
7. T. Ganelius, C. R., **242**, 6 (1956).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.