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1965

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Abstract

Full Text

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THE S. N. BERNSTEIN INTERPOLATION PROCESS IN THE COMPLEX DOMAIN

(Presented by Academician S. N. Bernstein on 24 II 1965)

MATHEMATICS

1°. Denote by C the set of all functions $f(x)$ continuous on the segment $[-1, 1]$. By $L_n(f, x)$ denote the Lagrange interpolation polynomial of degree n , constructed for the $(n + 1)$ rows of the matrix of nodes

$$\begin{array}{c} x_1^{(1)} \\ x_1^{(2)} x_2^{(2)} \\ \dots \end{array} \quad (1)$$

$$-1 \leq x_1^{(n)} < x_2^{(n)} < \dots < x_n^{(n)} \leq 1, \quad n = 1, 2.$$

The classical Bernstein-Faber theorem ^(1,2) asserts that there is no matrix of nodes of the form (1) for which, for every $f \in C$, the relation

$$L_n(f, x) \rightarrow f(x), \quad n \rightarrow \infty$$

holds uniformly.

For the case of Chebyshev nodes

$$x_k^{(n)} = \cos[(2n - 2k + 1)\pi/2n],$$

$k = 1, 2, \dots, n$, which in the theory of interpolation of functions of a real variable are in a certain sense the best possible, G. Grünwald ⁽³⁾ and I. Marcinkiewicz ⁽⁴⁾ constructed such an $f \in C$ that at every point $x \in [-1, 1]$ the equality

$$\overline{\lim}_{n \rightarrow \infty} L_n(f, x) = \infty$$

holds. In connection with the Bernstein-Faber theorem there naturally arose the question of replacing the Lagrange interpolation process by another interpolation process $\{A_n(f, x)\}_{n=1}^{\infty}$ which, already for every $f \in C$, uniformly satisfies the relation

$$A_n(f, x) \rightarrow f(x), \quad n \rightarrow \infty.$$

This problem was solved by S. N. Bernstein and L. Fejér ⁽⁵⁾. The solution of S. N. Bernstein ⁽⁶⁾ is remarkable in that it is obtained by a simple modification of the Lagrange interpolation formula, and the ratio of the degree of the interpolation polynomial $A_n(f, x)$ to the number of its nodes can be made arbitrarily close to one. The interpolation polynomials $A_n(f, x)$ of S. N. Bernstein were also studied in works ⁽⁷⁻¹⁰⁾. Until now the S. N. Bernstein interpolation polynomials have been studied only in the real domain. In this note they are studied in the complex domain.

Denote by A the set of all functions $f(z)$ analytic inside the disk $|z| < 1$ and continuous in the closed disk $|z| \leq 1$. Introduce in A the norm according to the equality

$$\|f\| = \max_{|z| \leq 1} |f(z)|.$$

Obviously, A is a Banach space.

For simplicity we shall consider the special case of the S. N. Bernstein interpolation polynomials $A_n(f, z)$ ⁽⁶⁾, when

$$A_n(f, z) = \sum_{k=1}^m f(z_{2k-1}^{(n)}) [l_{2k-1}^{(n)}(z) + l_{2k}^{(n)}(z)], \quad n = 2m, \quad (2)$$

where $\{l_j^{(n)}(z)\}_{j=1}^n$ are the fundamental Lagrange interpolation polynomials.

2°. First of all, by means of very simple estimates we shall prove the following theorem.

Theorem 1. The polynomials $A_n(f, z)$ constructed at the nodes

$$z_k^{(n)} = e^{i2k\pi/n}, \quad k = 1, 2, \dots, n; \quad n = 1, 2, \dots, \quad (3)$$

for $f \in A$, converge uniformly inside the circle $|z| < 1$ to $f(z)$.

Proof. It is obvious that

$$A_n(f, z) = 2 \sum_{k=1}^m f(z_{2k-1}^{(n)}) l_{2k-1}^{(n)}(z) + \sum_{k=1}^m f(z_{2k-1}^{(n)}) [l_{2k}^{(n)}(z) - l_{2k-1}^{(n)}(z)]. \quad (4)$$

It is easy to see that, for the nodes (3), the fundamental Lagrange interpolation polynomials $l_k^{(n)}(x)$ have the form

$$l_k^{(n)}(z) = \frac{(1 - z^n)z_k}{n(z_k - z)}, \quad k = 1, 2, \dots, n. \quad (5)$$

Consequently,

$$\sum_{k=1}^m f(z_{2k-1}^{(n)}) l_{2k-1}^{(n)}(z) = \sum_{k=1}^m f(e^{i2(2k-1)\pi/n}) \frac{1-z^n}{e^{i2(2k-1)\pi/n} - z} \frac{e^{i2(2k-1)\pi/n}}{n}.$$

Hence, as $n \rightarrow \infty$, by Cauchy's formula we obtain

$$\lim_{n \rightarrow \infty} \sum_{k=1}^m f(z_{2k-1}^{(n)}) l_{2k-1}^{(n)}(z) = \frac{1}{4\pi i} \int_{\Gamma} \frac{f(\zeta) d\zeta}{\zeta - z} = \frac{f(z)}{2}, \quad |z| < 1. \quad (6)$$

From equality (5) it is immediately seen that

$$|l_{2k}(z) - l_{2k-1}(z)| \leq \frac{1 + |z|^n}{n(1 - |z|)^2} |z_{2k} - z_{2k-1}|.$$

Therefore, since $\sum_{k=1}^m |z_{2k} - z_{2k-1}| \leq 2\pi$, we have

$$\left| \sum_{k=1}^m f(z_{2k-1}) [l_{2k}(z) - l_{2k-1}(z)] \right| \leq \frac{4\pi \max_{|z| \leq 1} |f(z)|}{n(1 - |z|)^2}. \quad (7)$$

As a consequence of (4), (6), (7) we have

$$\lim_{n \rightarrow \infty} A_n(f, z) = f(z), \quad |z| < 1.$$

By Vitali's theorem, the sequence $\{A_n(f, z)\}$ converges inside the circle $|z| < 1$ uniformly. From equality (5) it is seen that the assumption that n is even is not essential.

3°. Theorem 1 admits the following generalization. Let D be a bounded continuum of the z -plane such that its complement D_1 is a simply connected domain. We shall assume that the contour Γ of the domain D is a closed analytic Jordan curve. In fact, these conditions can be substantially weakened. By $w = \Phi(z)$ we denote the function mapping D_1 conformally onto the domain $|w| > 1$ of the w -plane under the condition $\Phi(\infty) = \infty$. Let $z = \psi(w)$ be the inverse function. As is known,

$$\psi(w) = cw + c_0 + c_1/w + \dots \quad (c \neq 0).$$

Let the points

$$z_1^{(n)}, z_2^{(n)}, \dots, z_n^{(n)}, \quad n = 1, 2, \dots, \quad (8)$$

be situated on the curve Γ . According to L. Fejér [11], the points (8) are called regularly situated on the curve Γ if the equalities

$$z_k^{(n)} = \psi(w_k^{(n)}), \quad w_k^{(n)} = e^{i2\pi k/n}, \quad k = 1, 2, \dots, n; \quad n = 1, 2, \dots$$

are satisfied.

Theorem 2. Let the points (8) be regularly situated on the curve Γ and let $f(z)$ be R -integrable on Γ . Then the polynomials (2) satisfy, for any point $z \in D$, the relation

$$A_n(f, z) \rightarrow \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta) d\zeta}{\zeta - z}, \quad n \rightarrow \infty. \quad (9)$$

The convergence is uniform in every closed domain situated inside D .

In the course of the proof of this theorem the following lemma was used:

Lemma. Under the hypotheses of Theorem 2, at every point $z \in D$,

$$\sum_{k=1}^m |l_{2k}^{(n)}(z) - l_{2k-1}^{(n)}(z)| \rightarrow 0, \quad n \rightarrow \infty, \quad n = 2m. \quad (10)$$

Relation (10) holds uniformly in every domain situated inside D .

For the proof of the lemma, let us note that

$$\begin{aligned} l_{2k}^{(n)}(z) - l_{2k-1}^{(n)}(z) &= \frac{\omega_n(z)}{c^n} \frac{1}{in(z - z_{2k})(z - z_{2k-1})} \times \\ &\times \left[z \left(\frac{\delta_{n,2k}}{\gamma_{n,2k}} - \frac{\delta_{n,2k-1}}{\gamma_{n,2k-1}} \right) + \left(z_{2k} \frac{\delta_{n,2k-1}}{\gamma_{n,2k-1}} - z_{2k-1} \frac{\delta_{n,2k}}{\gamma_{n,2k}} \right) \right]; \\ \gamma_{n,k} &= \frac{\omega'_n(z_k)}{inc^n} \delta_{n,k}; \quad \delta_{n,k} = \frac{d\psi(w_k^{(n)})}{d\varphi}; \end{aligned}$$

$$w = e^{i\varphi}; \quad \omega_n(z) = \prod_{k=1}^n (z - z_k^{(n)}),$$

and use the following known facts:

- 1) In the case of an analytic curve Γ , the modulus of the derivative of the mapping function $\Phi(z)$ is bounded on Γ above and below by positive constants ^(12,13).
- 2) For regularly distributed nodes ⁽¹⁴⁾

$$\omega_n(z)/(-c^n) \rightarrow 1, \quad n \rightarrow \infty. \quad (11)$$

Relation (11) holds uniformly in the domain $B \subset D$.

3) The relation ⁽¹⁵⁾

$$\gamma_{n,k} \rightarrow 1, \quad n \rightarrow \infty \quad (12)$$

holds uniformly with respect to k .

With the help of these facts and simple calculations, the lemma is obtained.

Proof of Theorem 2. By virtue of the lemma and equality (4),

$$\lim_{n \rightarrow \infty} A_n(f, z) = 2 \lim_{n \rightarrow \infty} \sum_{k=1}^m f(z_{2k-1}^{(n)}) l_{2k-1}^{(n)}(z), \quad z \in D. \quad (13)$$

But

$$\sum_{k=1}^m f(z_{2k-1}^{(n)}) l_{2k-1}^{(n)}(z) = \frac{1}{4\pi i} \sum_{k=1}^m \frac{f[\psi(w_{2k-1}^{(n)})]}{z - \psi(w_{2k-1}^{(n)})} \frac{\omega_n(z)}{c^n} \frac{\delta_{k,2k-1}}{\gamma_{n,2k-1}} \frac{4\pi}{n}.$$

Therefore, by virtue of relations (11) and (12), we obtain

$$\lim_{n \rightarrow \infty} \sum_{k=1}^m f(z_{2k-1}^{(n)}) l_{2k-1}^{(n)}(z) = \frac{1}{4\pi i} \int_{\Gamma} \frac{f(\zeta) d\zeta}{\zeta - z}. \quad (14)$$

From (13) and (14), (9) follows.

4°. On the boundary of the domain the polynomials (2) may diverge. This is seen from the theorem:

Theorem 3. *Suppose that the points*

$$z_k^{(n)} = e^{i\theta_k}, \quad \theta_k = (2k-1)\pi/n, \quad k = 1, 2, \dots, n; \quad n = 1, 2, \dots \quad (15)$$

are taken as nodes.

Then there exist such $f \in A$ for which the polynomials (2) diverge at the point $z = 1$.

We outline the proof. For the nodes (15) we have

$$A_n(f, 1) = \frac{2}{n} \sum_{k=1}^m f(z_{2k-1}^{(n)}) \left(\frac{z_{2k}^{(n)}}{z_{2k}^{(n)} - 1} + \frac{z_{2k-1}^{(n)}}{z_{2k-1}^{(n)} - 1} \right), \quad n = 2m.$$

Therefore it is sufficient to prove that $\lim_{n \rightarrow \infty} \lambda_n = \infty$, where

$$\lambda_n = \frac{2}{n} \sum_{k=1}^m \left| \frac{z_{2k}^{(n)}}{z_{2k}^{(n)} - 1} + \frac{z_{2k-1}^{(n)}}{z_{2k-1}^{(n)} - 1} \right|.$$

From this we obtain that

$$\lambda_n > \sum_{k=1}^{\lfloor (n-2)/4 \rfloor} \frac{1}{n \sin(4k-1)\pi/2n} - 1 > \frac{1}{2\pi} \ln n + O(1).$$

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Received
16 I 1965

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