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B. V. Medvedev, A. D. Sukhanov

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Abstract

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PHYSICS

B. V. MEDVEDEV, A. D. SUKHANOV

ON THE S -MATRIX IN THE HEISENBERG REPRESENTATION

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1. With the development of axiomatic methods in quantum field theory, representations of the S -matrix in the form of functional expansions in the operators of asymptotic fields $\varphi_{in}(x)$ (or $\varphi_{out}(x)$) have become widespread ^(1,2). At the same time, in ⁽³⁾, outside the framework of perturbation theory, the possibility was substantiated of representing the S -matrix in the form

$$S_{in} = T_W \exp \left\{ i \int_{-\infty}^{\infty} \mathcal{L}_{in}(z) dz \right\}, \quad (1)$$

where T_W is the Wick T -product ^(3,4), defined by the expansion according to Wick's theorem, and the operator $\mathcal{L}_{in}(z)$ has the meaning of an effective interaction Lagrangian (with counterterms) ⁽⁵⁾, expressed through normal products of the operators $\varphi_{in}(x)$ and their derivatives.

Of special interest is the derivation of the S -matrix directly in the Heisenberg representation, since the corresponding results (see, for example, ⁽⁶⁾) have not received due circulation. Below we propose a consistent method for obtaining the S -matrix in the Heisenberg representation, based on the assumption of the existence of the so-called "half" S -matrix*.

2. As the starting point it is natural to choose the in -representation of the S -matrix in the form (1), and, by the very meaning of the introduction of the S -matrix, it carries out ⁽⁶⁾ a unitary transformation to a new out -representation, for example, for the field operators

$$\varphi_{out}(x) = S_{in}^+ \varphi_{in}(x) S_{in}; \quad \varphi_{in}(x) = S_{in} \varphi_{out}(x) S_{in}^+, \quad (2)$$

and similarly for the others. Since we construct the theory on the basis of a number of general axioms ^(2,3), we shall obtain the "half" S -matrix $S_{in}(\sigma, -\infty)$ not in the usual way, by solving the Tomonaga-Schwinger equation, but directly from the matrix S_{in} , guided by the requirements: a) relativistic covariance; b)

independence of the specific choice of the spacelike surface σ ; c) finiteness; d) unitarity:

$$S_{in}^+(\sigma, -\infty)S_{in}(\sigma, -\infty) = S_{in}(\sigma, -\infty)S_{in}^+(\sigma, -\infty) = 1; \quad (3)$$

e) fulfillment of the group property:

$$S_{in} = S_{in}(\infty, \sigma)S_{in}(\sigma, -\infty), \quad (4)$$

f) boundary conditions:

$$\lim_{\sigma \rightarrow \infty} S_{in}(\sigma, -\infty) = S_{in}; \quad \lim_{\sigma \rightarrow -\infty} S_{in}(\sigma, -\infty) = 1. \quad (5)$$

The simplest possibility for obtaining a matrix $S_{in}(\sigma, -\infty)$, satisfying the enumerated requirements, from S_{in} of the form (1) consists in the following:

* The results were reported at the Fifth All-Union Conference on the Theory of Elementary Particles (Uzhgorod, October 1963).

perform the functional substitution $\mathcal{L}_{in}(z)$ by $\theta(\sigma(x) - z^0)\mathcal{L}_{in}(z)$. However, as has already been shown in perturbation theory ^(7,4), the half S -matrix obtained in this way will not satisfy requirements c), d), and e), if only $\mathcal{L}_{in}(z)$ contains time derivatives of fields higher than first, i.e., in all renormalized theories (counterterms!). At the same time, in ⁽⁴⁾ it was shown that the indicated difficulties in the definition of $S_{in}(\sigma, -\infty)$ can be avoided if one first passes, in the expression for S_{in} of the type (1), from the T_W -product to the T_D -product (Dyson T -product, defined by explicit chronological ordering ^(3,4)) and simultaneously from $\mathcal{L}_{in}(z)$ to $H_{in}(z; \sigma)$, the interaction Hamiltonian $H_{in}(z; \sigma)$ being obtainable, according to definite rules, from $\mathcal{L}_{in}(z)$ ^(4,8).

Since in the present case we do not have at our disposal a concrete expression for $\mathcal{L}_{in}(z)$, relying on the analogy with perturbation theory ⁽⁴⁾, we shall suppose that the matrix S_{in} of the form (1) can be represented in the form*

$$S_{in} = T_D \exp \left\{ -i \int_{-\infty}^{\infty} H_{in}(z; \sigma) dz \right\} = S_{in}[H_{in}(z; \sigma)], \quad (6)$$

where, generally speaking, $H_{in}(z; \sigma) \neq -\mathcal{L}_{in}(z)$.

Then the "half" S -matrix, if it exists at all, may be defined by the relation**

$$S_{in}(\sigma, -\infty) = S_{in}[\theta(\sigma(x) - z^0)H_{in}(z; \sigma')] = T_D \exp \left\{ -i \int_{-\infty}^{\sigma} H_{in}(z; \sigma') dz \right\}, \quad (7)$$

which, as is known from perturbation theory ⁽⁴⁾, in renormalizable theories can satisfy all requirements a)–e), in connection with which below, for simplicity, we shall put $\sigma(x) = x^0 = \text{const}$.

3. We shall now pass to the out-representation. By the general rule for transforming operators, one must have

$$S_{out} = S_{in}^+ S_{in} S_{in}, \quad (8)$$

and, since under the transformation (2) products are transformed in the same way, and consequently also polynomials and series of operators, it follows that

$$S_{out} = S_{in}[H_{out}(z)], \quad (9)$$

where $H_{out}(z) = S_{in}^+ H_{in}(z) S_{in}$, while the notation in (9) means that the functional dependence (in the present case the T_D -product) of S_{out} on $H_{out}(z)$ is exactly the same as that of S_{in} on $H_{in}(z)$ in (6). However, the meaning of the operator S_{out} is not yet clear. If, moreover, we take into account in (8) the unitarity of S_{in} , it turns out that

$$S_{out} = S_{in} = S, \quad (10)$$

i.e., the value of S_{out} coincides with the value of S_{in} .

At the same time,

$$S_{out}(x^0, -\infty) = S^+ S_{in}(x^0, -\infty) S = T_D \exp \left\{ -i \int_{-\infty}^{x^0} H_{out}(z) dz \right\}. \quad (11)$$

Thus, the matrix $S_{out}(x^0, -\infty)$ differs from the matrix $S_{in}(x^0, -\infty)$, although its functional dependence on $H_{out}(z)$ has remained

* The interaction Hamiltonian used by us should, more properly, be denoted $H_3^v(z; \sigma)$, since it is, generally speaking, distinct from the in-image of the interaction Hamiltonian ⁽⁴⁾. Our notation is intended to emphasize that this operator depends precisely on the fields $\varphi_{in}(z)$, and not on $\varphi_{out}(z)$ or on anything else. Further, since in our case the free operators are $\varphi_{in}(x)$, the matrix $S_{in}(\sigma, -\infty)$ has no relation to the evolution operator $U(\sigma, -\infty)$, but coincides with the matrix $V_+(\sigma)$ in the notation of ⁽⁹⁾.

** Such a definition, naturally, contains a factor of the form $\exp[iF_{in}(\sigma)]$, where $F_{in}(\sigma)$ is a Hermitian local operator which, from the point of view of the complete S -matrix, is immaterial.

same (T_D -product). Similarly, for the other “half” of the S -matrix we have

$$S_{out}(\infty, x^0) = S^+ S_{in}(\infty, x^0) S = S_{in}^+(x^0, -\infty) S. \quad (12)$$

However, the group property (4) is still preserved in the out-representation, i.e.

$$S_{out}(\infty, x^0) S_{out}(x^0, -\infty) = S^+ S_{in}(\infty, x^0) S S^+ S_{in}(x^0, -\infty) S = S. \quad (13)$$

Let us note that from (12) there follows a noteworthy formula, on which the subsequent transformations are based:

$$S_{in}(x^0, -\infty) S_{out}(\infty, x^0) = S. \quad (14)$$

Thus, for the complete scattering matrix there are three expressions in terms of “halves,” namely (4), (13), and (14).

Let us now introduce, in addition to the in- and out-representations, a collection of j -representations ($j = 1, \dots, n$), according to the formula

$$A_j(z) = S_{in}^+(x_j^0, -\infty) A_{in}(z) S_{in}(x_j^0, -\infty), \quad (15)$$

where $A_{in}(z)$ is an arbitrary in-operator. Then

$$S_j(\infty, x_j^0) = S_{in}^+(x_j^0, -\infty) S; \quad S_j(x_j^0, -\infty) = S_{in}(x_j^0, -\infty), \quad (16)$$

where the latter equality denotes not only the same functional dependence, but also coincidence of the operators.

Let us note that the matrix $S_j = S_j(\infty, x_j^0) S_j(x_j^0, -\infty)$ is not equal to the matrix S , but is related to it by the formula

$$S_j = S_{in}^+(x_j^0, -\infty) S S_{in}(x_j^0, -\infty), \quad (17)$$

i.e. it has only the same functional dependence. At the same time, from the first equality in (16) we obtain

$$S = S_{in}(x_j^0, -\infty) S_j(\infty, x_j^0). \quad (18)$$

Considering now simultaneously two such representations i and j (taking $i > j$), we obtain, with allowance for the fact that all transformations (15) form a group, that

$$A_i(z) = S_j^+(x_i^0, x_j^0) A_j(z) S_j(x_i^0, x_j^0), \quad (19)$$

where

$$S_j(x_i^0, x_j^0) = S_{in}^+(x_j^0, -\infty) S_{in}(x_i^0, x_j^0) S_{in}(x_j^0, -\infty). \quad (20)$$

Applying next the transformation (19) to the matrix $S_j(\infty, x_j^0)$, we obtain the relation

$$S_j(\infty, x_j^0) = S_j(x_i^0, x_j^0) S_i(\infty, x_i^0), \quad (21)$$

which, in combination with (18), gives for the complete S -matrix the expression

$$S = S_{in}(x_1^0, -\infty) S_j(x_i^0, x_j^0) S_i(\infty, x_i^0). \quad (22)$$

Carrying out these arguments now for all j successively, beginning with the first up to n , we arrive at

$$S = S_{in}(x_1^0, -\infty) \prod_{j=1}^{n-1} S_j(x_{j+1}^0, x_j^0) S_n(\infty, x_n^0). \quad (23)$$

Writing (18) for $j = n$ and comparing the expression obtained with (23), we also see that

$$S_{in}(x_n^0, -\infty) = S_{in}(x_1^0, -\infty) \prod_{j=1}^{n-1} S_j(x_{j+1}^0, x_j^0). \quad (24)$$

To pass to the expressions of interest to us, it is necessary in (23) and (24) to carry out the limiting transition $n \rightarrow \infty$ in such a way that, in doing so, $x_1^0 \rightarrow -\infty$, $\Delta_j = \max(x_{j+1}^0 - x_j^0) \rightarrow 0$, while $x_n^0 \rightarrow \infty$ in (23) and $x_n^0 \rightarrow x^0 = \text{const}$ in (24). Let us first note that the expression $S_{in}(x_{j+1}^0, x_j^0)$ entering into $S_j(x_{j+1}^0, x_j^0)$, according to (20), with account of (7), in the limit $\Delta_j \rightarrow 0$, if only terms of first order of smallness in Δ_j are retained, gives

$$\lim_{\Delta_j \rightarrow 0} S_{in}(x_{j+1}^0, x_j^0) = 1 - i \int_{x_j^0}^{x_{j+1}^0} H_{in}(z) dz. \quad (25)$$

Substituting now (25) into (20), and (20) into (23) and (24), we obtain for the S -matrix, in the indicated limit, an expression in the form of a Dyson antichronological exponential

$$S = \tilde{T}_D \exp \left\{ -i \int_{-\infty}^{\infty} \mathbf{H}_z(z) dz \right\} \quad (26)$$

of Hamiltonians taken, for each point z , in its own special representation, determined by the “half” S -matrix with the separating surface passing through this point z :

$$H_{(z)}(z) = S_{in}^+(z^0, -\infty) H_{in}(z) S_{in}(z^0, -\infty). \quad (27)$$

But it is easy to see that such an infinite collection of representations forms precisely the Heisenberg representation

$$H_z = S_{in}^+(z^0, -\infty) H_{in}(z) S_{in}(z^0, -\infty), \quad (28)$$

i.e. formula (26) means that

$$S = S_{in} = S_{out} = S_{\Gamma} = T_D \exp \left\{ -i \int_{-\infty}^{\infty} \mathbf{H}(z) dz \right\}; \quad (29)$$

and analogously from (24) we obtain

$$S_{in}(x^0, -\infty) = S_{\Gamma}(x^0, -\infty) = \tilde{T}_D \exp \left\{ -i \int_{-\infty}^{x^0} \mathbf{H}(z) dz \right\}. \quad (30)$$

Thus, in the transition from the in -representation to the Heisenberg one, the functional dependence in the S -matrix changes; namely, chronological ordering passes into antichronological ordering.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Moscow Institute
of Radio Electronics and Mining Electromechanics

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