

CONDITIONS FOR DISCRETENESS AND CONTINUITY OF THE SPECTRUM IN THE CASE OF A SELF-ADJOINT SYSTEM OF FIRST-ORDER DIFFERENTIAL EQUATIONS

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Abstract

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MATHEMATICS

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CONDITIONS FOR DISCRETENESS AND CONTINUITY OF THE SPECTRUM IN THE CASE OF A SELF-ADJOINT SYSTEM OF FIRST-ORDER DIFFERENTIAL EQUATIONS

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Let the operator D be generated by the differential expression

$$Dy = i\Lambda y' + Q(x)y \quad (1)$$

in the space $L_2(-\infty, +\infty)$ of vector-functions $y(x) = (y_1, \dots, y_k)$ with scalar product

$$[a, b] = \int_{-\infty}^{+\infty} (a, b) dx = \int_{-\infty}^{+\infty} \sum_{m=1}^k a_m(x) \overline{b_m(x)} dx.$$

It is assumed that in the principal part of the operator there stands a constant real diagonal invertible matrix, and that the matrix-function $Q(x) \equiv Q^*(x)$ is continuous. Under these conditions the operator D is self-adjoint and unbounded in both directions.

If the dimension of the unitary space E_k is even ($k = 2p$) and

$$\Lambda = \begin{pmatrix} 1_p & 0_p \\ 0_p & -1_p \end{pmatrix},$$

then, in essence, we are dealing with a Hamiltonian, i.e., with the operator

$$T = J \frac{d}{dx} + R(x) = UDU^{-1} \quad \text{for} \quad J = \begin{pmatrix} 0_p & 1_p \\ -1_p & 0_p \end{pmatrix}, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \cdot 1_p & i \cdot 1_p \\ 1_p & 1_p \end{pmatrix}, \quad (2)$$

which arises from the canonical system of equations $J(dy/dx) = H(x, \lambda)y$ in the case when the spectral parameter λ enters additively into the Hermitian matrix

$H(x, \lambda)$. For $p = 1$ the operator T was studied in a number of papers by E. C. Titchmarsh on the relativistic Dirac equation.

The main content of this note is the assertion that the operator D may have a purely discrete spectrum. This circumstance is specific to the case of a system of equations, since in the case of a single equation the spectrum fills the entire axis continuously (and thus the system (1) with diagonal $Q(x)$ has a continuous spectrum on the whole axis, since then D decomposes into k copies of scalar operators).

We first consider a special case which is of independent interest. Let $\Lambda^2 = 1_k$ and suppose that $Q'(x)$ exists. Then the operator D^2 is unitarily equivalent to an operator of Sturm-Liouville type. Indeed,

$$D^2 = -d^2/dx^2 + i(\Lambda Q + Q\Lambda)d/dx + (Q^2 + i\Lambda Q'),$$

since

$$(d/dx)Q = Q' + Q(d/dx).$$

Next,

$$A^{-1}D^2A = -d^2/dx^2 + A^{-1}[-2A' + i(\Lambda Q + Q\Lambda)A]d/dx + A^{-1}[-A'' + i(\Lambda Q + Q\Lambda)A' + (Q^2 + i\Lambda Q')A]$$

for any invertible twice differentiable matrix $A(x)$. The equation

$$A'(x) = (i/2)[\Lambda Q(x) + Q(x)\Lambda]A(x)$$

always has a unitary solution $U(x)$ (for example, the solution with $U(0) = 1_k$); therefore, finally, we obtain

$$U^{-1}D^2U = -d^2/dx^2 + U^{-1}(x)X(x)U(x),$$

where

$$X(x) = Q^2(x) - \frac{1}{4}(\Lambda Q + Q\Lambda)^2 + \frac{i}{2}(\Lambda Q' - Q'\Lambda).$$

Thus, we can conclude: if

$\Lambda = \pm 1_k$, then the spectrum of the operator D continuously fills the entire axis. If, however, $\Lambda \neq \pm 1_k$ (although $\Lambda^2 = 1_k$), then condition *

$$\lim_{|x| \rightarrow \infty} \mu X(x) = +\infty \quad (3)$$

guarantees the discreteness of the spectrum of the operator D .

In Theorem 1 a method is proposed for obtaining sufficient criteria for discreteness of the spectrum.

Theorem 1. 1) If, for some differentiable matrix $B(x) \equiv B^*(x)$,

$$\lim_{|x| \rightarrow \infty} \mu [B' - B\Lambda^{-2}B - 2\operatorname{Im}(B\Lambda^{-1}Q)] = +\infty, \quad (4)$$

then the operator D has discrete spectrum. For example, the spectrum is discrete if

$$\lim_{|x| \rightarrow \infty} \nu [\operatorname{Im}(B_0\Lambda^{-1}Q(x))] = -\infty \quad (5)$$

for some constant matrix $B_0 = B_0^*$.

2) On the other hand, if the spectrum is discrete, then necessarily the matrix $Q(x)$ is nondiagonal and satisfies the condition

$$\lim_{|x| \rightarrow \infty} \mu \int_x^{x+\omega} Q^2(t) dt = +\infty \quad (\omega \text{ fixed}), \quad (6)$$

and in the series of numbers $\lambda_1, \dots, \lambda_k$ standing on the diagonal of the matrix Λ , there must be at least one change of sign.

Let us illustrate all that has been said by the example of the operator

$$D_0 = i \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} p & q \\ \bar{q} & r \end{pmatrix}, \quad (7)$$

where $\lambda_1\lambda_2 \neq 0$, $q(x) = R(x) + iI(x)$. Let $\lambda_2 = -\lambda_1 = -1$ and let p' , r' , q' exist. Then condition (3) is rewritten in the following form:

$$\lim_{|x| \rightarrow \infty} [|q(x)|^2 - |q(p+r) + iq'|] = +\infty. \quad (8)$$

We give several sufficient criteria for discreteness of the spectrum of the operator D_0 , obtained with the aid of Theorem 1:

1) If $\lambda_2 = -\lambda_1$, then from (5) one can obtain the criterion

$$\lim_{|x| \rightarrow \infty} \left[\alpha R(x) + \beta I(x) - \frac{1}{2} |p(x) + r(x)| \right] = +\infty \quad (9)$$

for some α and β , where $\alpha^2 + \beta^2 = 1$. In the case of arbitrary λ_1, λ_2 this condition becomes less transparent:

$$\lim_{|x| \rightarrow \infty} \left[(\lambda_1 - \lambda_2)(\alpha R + \beta I) - \sqrt{(\lambda_1 + \lambda_2)^2 (\alpha R + \beta I)^2 + (p\lambda_2 - r\lambda_1)^2} \right] = +\infty \quad (10)$$

for some α, β ($\alpha^2 + \beta^2 = 1$).

2) If $R'(x)$ exists, then the condition

$$\lim_{|x| \rightarrow \infty} [-2\lambda_1 \lambda_2 R^2(x) - |\lambda_1 - \lambda_2| |R(p\lambda_2 - r\lambda_1) + i\lambda_1 \lambda_2 R'|] = +\infty \quad (11)$$

is sufficient for discreteness of the spectrum of the operator D_0 . For $\lambda_2 = -\lambda_1$ one can additionally obtain the condition

$$\lim_{|x| \rightarrow \infty} [\pm \sqrt{2} R(x) I(x) - |R(p + r) + i\lambda_1 R'|] = +\infty. \quad (12)$$

We note that the last two assertions remain valid if in them one formally replaces $R(x)$ by $I(x)$, and $I(x)$ by $R(x)$, since the passage from D_0 to the operator UD_0U^{-1} with the unitary matrix

$$U = \begin{pmatrix} 0 & \exp(ia_0) \\ \exp(ib_0) & 0 \end{pmatrix}$$

can interchange, in particular, $R(x)$ and $I(x)$.

* μN , νN and $\|N\|$ will denote, respectively, the least and greatest proper values and the Euclidean norm of

3) If $R'(x)$ and $I'(x)$ exist simultaneously, then each of the following two conditions guarantees discreteness of the spectrum:

$$\lim_{|x| \rightarrow \infty} \left[-\sqrt{2\lambda_1 \lambda_2} (R \pm I)^2 - |\lambda_1 - \lambda_2| ((R \pm I)(p\lambda_2 - r\lambda_1) + i\lambda_1 \lambda_2 (R' \pm I')) \right] = +\infty, \quad (13)$$

$$\lim_{|x| \rightarrow \infty} [-2\lambda_1 \lambda_2 |q(x)|^2 - |\lambda_1 - \lambda_2| |q(\lambda_2 p - \lambda_1 r) + i\lambda_1 \lambda_2 q'|] = +\infty. \quad (14)$$

In particular, from (14), for $\lambda_2 = -\lambda_1 = -1$, condition (8) is obtained, and without the requirement of differentiability of $p(x)$ and $r(x)$.

Proof of Theorem 1. Using the splitting principle ((1), §§ 1, 2), it is not hard to show that the spectrum of the operator D is discrete if and only if $\lim_{r \rightarrow +\infty} d(r) = +\infty$, where

$$d(r) = \inf_{y \in K(r, \infty)} \frac{\|Dy\|^2}{\|y\|^2} = \inf_y \frac{\int_{|x| \geq r} [|\Lambda y'|^2 + |Qy|^2 - 2 \operatorname{Re}(iQy, \Lambda y')] dx}{\int_{|x| \geq r} |y|^2 dx}. \quad (15)$$

Here $K(r, \infty)$ denotes the class of finite piecewise-smooth vector functions, the compact support of each of which is situated outside the interval $(-r, +r)$. Since always

$$\int_a^b (\Phi' f, g) dx = (\Phi f, g) \Big|_a^b - \int_a^b (\Phi f', g) dx - \int_a^b (\Phi f, g') dx,$$

then for any $y(x) \in K(r, \infty)$ and any self-adjoint differentiable matrix $B(x)$,

$$0 = \int_{|x| \geq r} [(B' y, y) + 2 \operatorname{Re}(B y, y')] dx. \quad (16)$$

Add expression (16) to the numerator in (15), and estimate the result from below as follows:

$$\begin{aligned} \int_{|x| \geq r} |Dy|^2 dx &= \int_{|x| \geq r} \{|\Lambda y'|^2 + (Q^2 y, y) + (B' y, y) + 2 \operatorname{Re}(P y, \Lambda y')\} dx \geq \\ &\geq \int_{|x| \geq r} (C y, y) dx \geq \left[\inf_{|x| \geq r} \mu C(x) \right] \int_{|x| \geq r} |y|^2 dx, \end{aligned}$$

where

$$P(x) = \Lambda^{-1} B - iQ; \quad C(x) = Q^2 + B' - P^* P = B' - B \Lambda^{-2} B - 2 \operatorname{Im}(B \Lambda^{-1} Q).$$

In this estimate we have applied the Cauchy-Bunyakovsky inequality:

$$2 \operatorname{Re}(P y, \Lambda y') \geq -2|P y| |\Lambda y'| \geq -|\Lambda y'|^2 - |P y|^2.$$

Condition (4) is proved.

Assertion (6) is a very particular case of the general fact which is formulated here for the one-dimensional case.

Lemma. Consider in $L_2(-\infty, +\infty)$ the operator

$$L = \sum_{i=0}^l Q_i(x) \frac{d^i}{dx^i},$$

where the matrices $Q_i(x)$ are continuous. If its resolvent R_λ is completely continuous, then

$$\lim_{|x| \rightarrow \infty} \mu \int_x^{x+\omega} \sum_i Q_i^* Q_i dt = +\infty \quad (\omega \text{ fixed}). \quad (17)$$

Remark. If among the Q_i constant matrices occur, then one may suppose that they do not enter under the summation sign in (17), since the inequality

$$\mu A_0 + \mu B(x) \leq \mu[A_0 + B(x)] \leq \nu A_0 + \mu B(x)$$

is valid.

For the proof of the lemma assume that (17) is not fulfilled. Then there exist numbers $|x_s| \rightarrow \infty$ and a sequence of constant vectors $|\xi_s| = 1$, for which

$$\int_{x_s}^{x_s+\omega} \sum_i (Q_i^* Q_i \xi_s, \xi_s) dt < C_0$$

for some ...

$\omega_0 > 0$, $c_0 > 0$. Fix an arbitrary smooth function $u_0(x)$, equal to zero outside $[0, \omega_0]$. Then the sequence of vector-functions $y_s(x) = u_0(x - x_s)\xi_s$ is noncompact in $L_2(-\infty, +\infty)$, whereas

$$\begin{aligned} \|Ly_s - z_0 y_s\|^2 &\leq 2\|z_0 y_s\|^2 + C_1 \sum_i \|Q_i y_s^{(i)}\|^2 = \\ &= C_2 + C_1 \sum_i \int_{x_s}^{x_s+\omega_0} |u_0^{(i)}|^2 |Q_i \xi_s|^2 dt \leq C_2 + C_3 \sum_i \int_{x_s}^{x_s+\omega_0} |Q_i \xi_s|^2 dt \leq C_4, \end{aligned}$$

which contradicts the complete continuity of the operator R_{z_0} . The lemma is proved.

Suppose that the initial operator D has discrete spectrum. "Normalize" the matrix Λ , i.e., pass from D to the operator $\tilde{D} = i\tilde{\Lambda} d/dx + \tilde{Q}(x) = |\Lambda|^{-1/2} D |\Lambda|^{-1/2}$, for which $\tilde{\Lambda}^2 = 1_k$ and $\tilde{Q}^* \equiv \tilde{Q}$. In this case the spectrum remains discrete, since the equation $\tilde{D}y = \lambda y$ can be rewritten in the form $i\Lambda z' + Qz = \lambda |\Lambda|z$ for $z = |\Lambda|^{-1/2}y$, which corresponds to considering the operator D in the metric

$$\int_{-\infty}^{+\infty} (|\Lambda|y, y) dx,$$

and it is equivalent to the original one. Since the diagonal entries of the matrix $\tilde{\Lambda}$ are the numbers $\lambda_i/|\lambda_i|$, there can be only two possibilities: either among the numbers $\Lambda_1, \dots, \Lambda_k$ there is at least one change of sign, or $\tilde{\Lambda} = \pm 1_k$. Since from the very beginning we could assume $Q(x)$ differentiable (the discreteness of the spectrum, by virtue of (15), is not destroyed by arbitrary additive bounded perturbations of the matrix $Q(x)$), it follows that for $\tilde{\Lambda} = \pm 1_k$ the spectrum is continuous on the whole axis, independently of the behavior of $\tilde{Q}(x)$, as was shown above. Theorem 1 is completely proved.

We now give several conditions guaranteeing the continuity of the spectrum independently of the properties of the matrix Λ .

Theorem 2. 1) *The spectrum of the operator D is continuous on the whole axis if at least one of the following three conditions is satisfied: for every $\delta > 0$*

$$\int_{M_\delta} \|Q(t)\|^2 dt < \infty, \quad (18)$$

where

$$M_\delta = \{x : \|Q'(x)\| > \delta\};$$

on some sequence of intervals Δ_i of unboundedly increasing length $|\Delta_i|$

$$\lim_{i \rightarrow \infty} \frac{1}{|\Delta_i|} \int_{\Delta_i} \|Q(t)\|^2 dt = 0; \quad (19)$$

for some (and hence for every) $\omega > 0$

$$\lim_{|x| \rightarrow \infty} \left\| \int_x^{x+\omega} Q^2(t) dt \right\| = 0. \quad (20)$$

2) *If one denotes*

$$\Omega = \limsup_{|x| \rightarrow \infty} \nu Q(x) - \liminf_{|x| \rightarrow \infty} \mu Q(x),$$

then the length of each gap in the continuous spectrum does not exceed Ω .

Theorem 2 is proved by the method of I. M. Glazman ⁽¹⁾, §31; to obtain (20) one uses inequality (15) from ⁽²⁾.

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CITED LITERATURE

¹ I. M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, 1963. ² R. V. Martynov, *Differential Equations*, 1, No. 12 (1965).

Note: Figure translations are in progress. See original paper for figures.

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