

On the Trace of a Differential Operator

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Abstract

Full Text

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ON THE TRACE OF A DIFFERENTIAL OPERATOR

(Presented by Academician I. G. Petrovskii, 12 III 1965)

Mathematics

Let two differential operators be given

$$S_1 = \frac{d^n}{dx^n}, \quad S_2 = \frac{d^n}{dx^n} + q(x), \quad (1)$$

defined on the interval $[0, 1]$ by identical regular boundary conditions (definition of regularity ⁽⁵⁾)

$$U_j(y) = U_{j0}(y) + U_{j1}(y) = \alpha_j y^{(k_j)}(0) + \dots + \beta_j y^{(k_j)}(1) + \dots, \quad j = 1, \dots, n, \quad (2)$$

where $q(x)$ is a sufficiently smooth function and

$$\int_0^1 q(x) dx = 0. \quad (3)$$

In the present work conditions are obtained for the regularization (existence) of the trace of the operator S_2

$$\sum_{k=1}^{\infty} \lambda_{2,k} = \sum_{k=1}^{\infty} (\lambda_{2,k} - \lambda_{1,k} - \dots), \quad (4)$$

where $|\lambda_{2,1}| \leq |\lambda_{2,2}| \leq \dots$ and $|\lambda_{1,1}| \leq |\lambda_{1,k}| \leq \dots$ are the eigenvalues of the operators S_2 and S_1 .

For $n = 2$, first in the work of I. M. Gelfand and B. M. Levitan ^(1,2), and then by other authors ^(3,4), the formula

$$\sum_{k=1}^{\infty} \lambda_{2,k} = \sum_{k=1}^{\infty} (\lambda_{2,k} - \lambda_{1,k}) = -\frac{q(1) + q(0)}{4}$$

was obtained, but in all these works the operators S_1 and S_2 were self-adjoint. The method used in the present work is based on the relation between the resolvents $R_{\lambda,1}$ and $R_{\lambda,2}$ of the operators S_1 and S_2

$$R_{\lambda,2} = R_{\lambda,1} - R_{\lambda,1} \cdot (qR_{\lambda,1}), \quad (5)$$

which is also valid in the non-self-adjoint case. This method makes it possible to prove the following result.

Let ω_i be the roots of the n -th degree of -1 ; $\rho^n = -\lambda$, and A_0 and A_1 constants defined as follows:

1. $n = 2\mu$.

$$A_0 = \frac{1}{n\theta_1} \sum_{i=1}^{\mu} \sum_{j=\mu+1}^{\mu} \bar{A}_{ij} \frac{\omega_j}{\omega_i - \omega_j}; \quad (6)$$

$$A_1 = \frac{1}{n\theta_1} \sum_{i=\mu+1}^n \sum_{j=1}^{\mu} A_{ij} \frac{\omega_j}{\omega_i - \omega_j}. \quad (7)$$

2. $n = 2\mu + 1$.

$$A_0 = \frac{1}{2n} \left[\frac{1}{\theta_2} \sum_{i=1}^{\mu} \sum_{j=\mu+1}^n \frac{\bar{A}_{ij}\omega_j}{\omega_i - \omega_j} + \frac{1}{\theta_3} \sum_{i=1}^{\mu+1} \sum_{j=\mu+2}^n \frac{\bar{A}_{ij}\omega_j}{\omega_i - \omega_j} \right]; \quad (8)$$

$$A_1 = \frac{1}{2n} \left[\frac{1}{\theta_2} \sum_{i=\mu+1}^n \sum_{j=1}^{\mu} \frac{\bar{A}_{ij}\omega_j}{\omega_i - \omega_j} + \frac{1}{\theta_3} \sum_{i=\mu+2}^n \sum_{j=1}^{\mu+1} \frac{\bar{A}_{ij}\omega_j}{\omega_i - \omega_j} \right]. \quad (9)$$

Here

$$\theta_1 = \begin{vmatrix} \alpha_1 \omega_1^{k_1} & \cdots & \alpha_1 \omega_{\mu}^{k_1} & \beta_1 \omega_{\mu+1}^{k_1} & \cdots & \beta_1 \omega_n^{k_1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n \omega_1^{k_n} & \cdots & \alpha_n \omega_{\mu}^{k_n} & \beta_n \omega_{\mu+1}^{k_n} & \cdots & \beta_n \omega_n^{k_n} \end{vmatrix}; \quad (10)$$

$$\theta_2 = \begin{vmatrix} \alpha_1 \omega_1^{k_1} & \cdots & \alpha_1 \omega_{\mu}^{k_1} & \beta_1 \omega_{\mu+1}^{k_1} & \cdots & \beta_1 \omega_n^{k_1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n \omega_1^{k_n} & \cdots & \alpha_n \omega_{\mu}^{k_n} & \beta_n \omega_{\mu+1}^{k_n} & \cdots & \beta_n \omega_n^{k_n} \end{vmatrix}; \quad (11)$$

$$\theta_3 = \begin{vmatrix} \alpha_1 \omega_1^{k_1} & \cdots & \alpha_1 \omega_{\mu+1}^{k_1} & \beta_1 \omega_{\mu+2}^{k_1} & \cdots & \beta_1 \omega_n^{k_1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_n \omega_1^{k_n} & \cdots & \alpha_n \omega_{\mu+1}^{k_n} & \beta_n \omega_{\mu+2}^{k_n} & \cdots & \beta_n \omega_n^{k_n} \end{vmatrix}^*, \quad (12)$$

where $\bar{A}_j^i$, $\bar{A}_{ij}$, and $\bar{A}_{ij}$ differ from the determinants standing in the denominator in that, in the i -th column, ω_i is replaced by ω_j . Then the following holds.

Theorem 1. *For $n \geq 2$ and regular boundary conditions, the trace S_2 exists and is equal to*

$$\sum_{k=1}^{\infty} \lambda_{2,k} = \sum_{k=1}^{\infty} (\lambda_{2,k} - \lambda_{1,k}) = A_1 q(1) - A_0 q(0). \quad (13)$$

Proof. Since the resolvents $R_{\lambda,1}$ and $R_{\lambda,2}$ are Hilbert-Schmidt operators with kernels $R_1(x, t; \lambda)$ and $R_2(x, t; \lambda)$, applying Lidskii's theorem (6)***, we obtain

$$\text{sp } R_{\lambda,1} = \sum_{k=1}^{\infty} \frac{1}{\lambda_{1,k} - \lambda} = \int_0^1 R_1(x, x; \lambda) dx; \quad (14)$$

$$\text{sp } R_{\lambda,2} = \sum_{k=1}^{\infty} \frac{1}{\lambda_{2,k} - \lambda} = \int_0^1 R_2(x, x; \lambda) dx. \quad (15)$$

Iterating (5) m times and taking sp, we obtain

$$\text{sp } R_{\lambda,2} - \text{sp } R_{\lambda,1} = \frac{d}{d\lambda} \left[\sum_{l=1}^m J_l(\lambda) \right] + R_m(\lambda), \quad (16)$$

where

$$J_l(\lambda) = \frac{(-1)^l}{l} \int_0^1 \cdots \int_0^1 q(x_1) \cdots q(x_l) R_1(x_1, x_2; \lambda) \cdots \\ \cdots R_l(x_l, x_1; \lambda) dx_1 \cdots dx_l. \quad (17)$$

* Comparing (11), (12), and (13) with the formulas in (5), p. 51, we see that the non-vanishing of θ_1, θ_2 , and θ_3 is equivalent to the regularity of the boundary conditions.

** For differential operators of a more general type this was done in (7).

Bearing in mind the simple identity

$$\frac{1}{2\pi i} \oint_{|\lambda|=r} \lambda \frac{d}{d\lambda} \left[\frac{1}{\pi(\lambda_i - \lambda)^{n_i}} \right] d\lambda = -\frac{1}{2\pi i} \oint \frac{d\lambda}{\prod_{i=1}^l (\lambda_i - \lambda)^{n_i}}, \quad (18)$$

multiplying (16) by $\lambda/2\pi i$ and integrating over the contour $|\lambda| = r$, we obtain

$$\sum_{|\lambda| < r} (\lambda_{2,k} - \lambda_{1,k}) = -\frac{1}{2\pi i} \sum_{l=1}^m \oint_{|\lambda|=r} J_l d\lambda + \widetilde{R}_m(r) = -\sum I_l(r) + \widetilde{R}_m(r). \quad (19)$$

According to (5), p. 71, in the regular case we have

$$|R_1(x, t; \lambda)| = O(1/|\rho|^{n-1}), \quad |J_l(\lambda)| = O(1/\rho^{l(n-1)}). \quad (20)$$

But condition (3) and the smoothness of $q(x)$ make it possible to increase the order of decrease by 1, i.e.

$$|J_l(\lambda)| = O(1/\rho^{l(n-1)+1}) = O(1/\rho^{(l-1)(n-1)+n}), \quad (21)$$

from which it follows that in (19) all terms, beginning with the second, vanish*. Therefore it is necessary to investigate $I_{1,1}(r)$, i.e. $J_1(\lambda)$.

From the formula given in (5), p. 37, by simple algebraic operations we obtain

$$R_1(x, t; \lambda) = \begin{cases} \frac{1}{n\rho^{n-1}\Delta(\rho)} \sum_{i,j=1}^n A_{ij}(\rho) \omega_j e^{\rho(\omega_i x - \omega_j t)}, & x > t, \\ \frac{1}{n\rho^{n-1}\Delta(\rho)} \sum_{i,j=1}^n B_{ij}(\rho) \omega_j e^{\rho(\omega_i x - \omega_j t)}, & x < t, \end{cases} \quad (22)$$

where

$$\Delta(\rho) = \det \|U_i(e^{\rho\omega_j x})\|_{i,j=1}^n,$$

and the determinants $A_{ij}(\rho)$ and $B_{ij}(\rho)$ differ from $\Delta(\lambda)$ in that in place of the i -th column there stand, respectively, the columns

$$U_{10}(e^{\rho\omega_j x})|_{l=1}^n \quad \text{and} \quad U_{11}(e^{\rho\omega_j x})|_{l=1}^n.$$

Substituting (22) into the expression for $I_1(r)$, passing to the variable $\lambda = -\rho^n$, and integrating by parts once in the inner integral, we obtain

$$I_1(r) = \frac{1}{2\pi i} \oint_{\delta_n} \sum_{i \neq j}^n \frac{A_{ij}(\rho)}{\Delta(\rho)} \frac{\omega_j}{\omega_i - \omega_j} e^{(\omega_i - \omega_j)\rho} d\rho \cdot q(1) - \frac{1}{2\pi i} \oint_{\delta_n} \sum_{i \neq j}^n \frac{A_{ij}(\rho)}{\Delta(\rho)} \frac{\omega_j}{\omega_i - \omega_j} d\rho \cdot q(0). \quad (23)$$

Here δ_n is an arc on the circle $|\rho| = r^{1/n}$ of angular measure $2\pi/n$. We choose δ_n so that the following conditions are satisfied:

1. If $n = 2\mu$, one can number the ω_i so that

$$\operatorname{Re}(\rho\omega_i) < 0, \quad i = 1, \dots, \mu,$$

$$\operatorname{Re}(\rho\omega_i) > 0, \quad i = \mu + 1, \dots, n, \quad \rho \in \delta_n. \quad (24)$$

* By choosing m sufficiently large, one can, according to (20), obtain that $|\widehat{R}_m(r)| = o(1)$.

2. For $n = 2\mu + 1$, one can divide δ_n into two arcs δ'_n and δ''_n of the same angular measure π/n , and number the ω_i in such a way that

$$\begin{aligned} \operatorname{Re}(\rho\omega_i) &< 0, & i = 1, \dots, \mu, \\ \operatorname{Re}(\rho\omega_i) &> 0, & i = \mu + 1, \dots, n, \end{aligned} \quad \rho \in \delta'_n, \quad (25)$$

$$\begin{aligned} \operatorname{Re}(\rho\omega_i) &< 0, & i = 1, \dots, \mu + 1, \\ \operatorname{Re}(\rho\omega_i) &> 0, & i = \mu + 2, \dots, n, \end{aligned} \quad \rho \in \delta''_n. \quad (26)$$

Using (22), (24), (25), and (26) for large $|\rho|$, we obtain, for example ($n = 2\mu$):

$$\frac{A_{ij}(\rho)}{\rho\Delta(\rho)} = \frac{A_{ij}}{\theta_1(\rho)} + o\left(\frac{e^{\rho\gamma_1}}{\rho}\right), \quad \operatorname{Re}(\rho\gamma_1) < 0, \quad \rho \in \delta_n, \quad i \leq \mu, \quad j \geq \mu + 1, \quad (27)$$

and in the opposite case

$$\frac{A_{ij}(\rho)}{\rho\Delta(\rho)} = O\left(\frac{e^{\rho\gamma_2}}{\rho}\right), \quad \operatorname{Re}(\rho\gamma_2) < 0, \quad \rho \in \delta_n.$$

Repeating analogous computations for the remaining terms of (23), substituting the obtained expressions into (23), and applying Jordan's lemma ⁽⁸⁾, we obtain (13).

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