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Abstract

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MATHEMATICS

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ON THE ADDITIVITY PRINCIPLE FOR CONSTRUCTING ECONOMICAL DIFFER- ENCE SCHEMES

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In all works (¹⁻⁷) and others devoted to economical (in the number of operations) methods for the numerical solution of multidimensional problems of mathematical physics, the representability of a multidimensional elliptic operator in the form of a sum of one-dimensional operators is used in one form or another. In (⁵⁻⁷), for equations and systems of equations of parabolic and hyperbolic types, locally one-dimensional schemes (l.o.s.) were studied, constructed on the basis of the principle of modeling a multidimensional differential equation by means of a system of one-dimensional equations, with subsequent application, for the solution of each of the one-dimensional equations, of the simplest difference schemes. Below this principle of constructing locally one-dimensional schemes is applied to an abstract Cauchy problem in a Banach space (the additivity principle).

1. Let a linear (unbounded) operator $A(t)$, depending on the parameter $t \in [0, T]$, be defined in a Banach space H , with a domain of definition $D(A)$ dense in H and independent of t , and with range $\Delta(A) \subset H$. Consider the abstract Cauchy problem

$$du/dt + A(t)u = f(t), \quad 0 \leq t \leq T, \quad u(0) = u_0, \quad u_0 \in D(A), \quad (1)$$

where the derivative is understood in the strong sense; $u = u(t) \in H$ (the desired function) and $f = f(t) \in H$ (the prescribed function) are defined on the interval $0 \leq t \leq T$. The boundary conditions are taken into account by the requirement $u = u(t) \in D(A)$. We shall be interested in the case when A is a sum of linear operators A_α with the same domains of definition and ranges as the operator A :

$$A(t) = \sum_{\alpha=1}^m A_\alpha(t). \quad (2)$$

We shall agree to call $A_\alpha(t)$ one-dimensional operators, and the corresponding Cauchy problems (1_α) with $A = A_\alpha$ one-dimensional problems. We shall assume that problem (1) is uniformly well posed.

2. Introduce the mesh $\omega_\tau = \{t^j = j\tau; j = 0, 1, \dots, [T/\tau]\}$ with step τ . The “multidimensional” problem (1) will be modeled (approximated) by means of m one-dimensional problems.

On each interval $[t^j, t^{j+1}]$, instead of (1), we shall successively solve the system of one-dimensional differential equations

$$dv_{(\alpha)}/dt + A_\alpha(t)v_{(\alpha)}(t) = f_\alpha(t),$$

$$t \in (t^j, t^{j+1}], \quad j = 0, 1, \dots; \quad \alpha = 1, 2, \dots, m, \quad (3)$$

with initial conditions

$$v_{(\alpha)}(t^j) = v_{(\alpha-1)}(t^{j+1}), \quad \alpha > 1; \quad v_{(1)}(t^j) = v_{(m)}(t^j) = v(t^j);$$

$$j \geq 0, \quad v(0) = u_0. \quad (4)$$

By the solution of problem (3)–(4) on the mesh ω_τ we shall mean the function

$$v = v^{j+1} = v_{(m)}(t^{j+1}).$$

The functions $f_\alpha = f_\alpha(t, \tau)$ belong to the domain of definition of $f(t)$ in H , depend, generally speaking, on τ , and satisfy the normalization condition

$$\left\| \sum_{\alpha=1}^m f_\alpha - f(t) \right\| = O(\tau^{k_0}), \quad k_0 > 0,$$

where $\|\cdot\|$ is the norm in H . In (5) we assumed that

$$\sum_{\alpha=1}^m f_\alpha = f.$$

The functions $v_{(1)}^{j+1}, \dots, v_{(m)}^{j+1}$ may be interpreted as successive approximations to the solution of problem (1), the number of which m is equal to the number of terms in the sum (2).

3. Let $U(t, t')$ be the resolving operator of equation (1) for $0 \leq t' \leq t \leq T$; let $U_\alpha(t, t')$ be the resolving operator of the one-dimensional equation (3_α) with number α . Replacing equation (1) by the locally one-dimensional system (3) means that $U(t, t')$ is approximated by the product

$$\widetilde{U}(t, t') = U_m(t, t') \dots U_1(t, t'),$$

so that $U \approx \widetilde{U}$. Thus the proposed method of one-dimensional modeling of problem (1) corresponds to an approximate factorization, or splitting, of the resolving operator $U(t, t')$ into one-dimensional operators. If $\widetilde{U} = U$, then for $f = 0$ we have $v^j = w^j$ for all $j = 0, 1, \dots$. If $j \neq 0$, then v^j coincides with w^j on the grid ω_t only under a special choice of the right-hand sides f_α , namely: $f_\alpha = 0$ for $\alpha = 1, 2, \dots, m-1$,

$$f_m(\theta) = w_{(m-1)}(t^{j+1}, \theta), \quad t^j \leq \theta \leq t^{j+1},$$

where $w_{m-1}(t, \theta)$ is the solution of the problem

$$\frac{dw_{(\alpha)}}{dt}(t, \theta) + A_\alpha(t)w_{(\alpha)}(t, \theta) = 0, \quad t^j \leq \theta \leq t \leq t^{j+1},$$

$$w_{(\alpha)}(\theta, \theta) = w_{(\alpha-1)}(t^{j+1}, \theta), \quad \alpha = 2, \dots, m-1,$$

$$w_{(1)}(\theta, \theta) = f(\theta).$$

Let us give the simplest examples when $\widetilde{U} = U$: 1) the heat-conduction equation with coefficients allowing separation of variables in a parallelepiped ($0 \leq x_\alpha \leq l_\alpha$, $\alpha = 1, 2, \dots, m$) with homogeneous boundary conditions; 2) a first-order hyperbolic-type equation with constant coefficients, etc. (V. I. Gol' din, N. N. Kalitkin). In general, if $A_\alpha(t_1)A_\beta(t_2) = A_\beta(t_2)A_\alpha(t_1)$, $\beta \neq \alpha$, $t_1, t_2 \in [0, T]$, then $U = \widetilde{U}$.

4. We shall say that problem (3), (4) is uniformly well-posed if there exist resolving operators $U_\alpha(t, t')$ for all $0 \leq t' \leq t \leq T$, $\alpha = 1, 2, \dots, m$, strongly continuous in the aggregate t, t' , and

$$\|U_\alpha(t, t')\| \leq 1 + M_D(t - t'),$$

where $M_D = \text{const} > 0$ does not depend on τ .

Theorem 1. *Let problem (3), (4) be uniformly well-posed and let the solution $u = u(t)$ of problem (1) satisfy the conditions: 1)*

$$\left\| (U_\alpha(t, t') - E) \frac{du}{dt} \right\| \leq \rho_1(\tau),$$

2)

$$\|(U_\alpha(t, t') - E)\psi_\alpha\| \leq \rho_2(\tau), \quad \psi_\alpha(t) = f_\alpha(t) - A_\alpha(t)u - \frac{du}{dt},$$

3)

$$\Psi_\alpha(t) = \int_{t_j}^t U_\alpha(t, \theta) \psi_\alpha(\theta) d\theta, \quad t \geq t_j,$$

is the solution of the inhomogeneous equation (3_α) with number α , with right-hand side $\psi_\alpha(t)$ and initial condition $\psi_\alpha(t_j) = 0$, $\alpha = 1, \dots, m$. Then the solution of problem (3), (4) converges uniformly in t to the solution of problem

$$\|v^j - u^j\| \leq \rho_3(\tau), \quad j = 1, 2, \dots$$

(here $\rho_k(\tau) \rightarrow 0$, $k = 1, 2, 3$, uniformly in t).

5. For solving problem (3)–(4), depending on the concrete form of A_α , one may use various suitable methods (for example, seek an exact expression for $v_{(\alpha)}$, use the method of characteristics, the method of straight lines, the method of finite differences, etc.).

Following (8), we shall seek an approximate solution of problem (3), (4) in a Banach space H_N (for example, in the space of functions defined on a grid ω_N in some domain of p -dimensional space $x = (x_1, \dots, x_p)$). Let P_N be a linear operator projecting H onto H_N ($P_N u = u_N \in H_N$, if $u \in H$) and $\lim_{N \rightarrow \infty} \|P_N u\|_N = \|u\|$ for every

$u \in H$, where $\|\cdot\|_N$ is the norm in H_N , $\|\cdot\|$ is the norm in H .

Replacing each one-dimensional equation (3_α) of number α by a difference scheme in H_N , we obtain a L.o.s. approximating the original problem (1). We shall consider two-level schemes. To equation (3_α) there corresponds in H_N a two-level scheme $R_\alpha^N y_{(\alpha)} = S_\alpha^N y_{(\alpha)} + \tau \varphi_\alpha$, where $y_{(\alpha)} = y_{(\alpha)}^{j+1}$, $y_{(\alpha)} = y_{(\alpha)}^j$, and R_α^N and S_α^N are linear operators depending on t, τ, N and mapping H_N into itself (see (8)). Taking into account the initial conditions $y_{(\alpha)} = y_{(\alpha-1)}$ for $\alpha > 1$ and $y_{(1)} = y_{(0)} y = y^j$, we obtain the L.o.s. in the form:

$$\begin{aligned} R_\alpha^N y_{(\alpha)} &= S_\alpha^N y_{(\alpha-1)} + \tau \varphi_\alpha, \quad \alpha = 1, 2, \dots, m, \quad t \in \omega_\tau, \quad y_{(0)} = y, \\ y(0) &= P_N u_0. \end{aligned} \quad (5)$$

The solution of this problem is the grid function $y = y^{j+1} = y_{(m)}^{j+1}$.

6. Problem (5) is posed correctly (the L.o.s. (5) is correct) if, for sufficiently large $N \geq N_0$ and sufficiently small $\tau \leq \tau_0$, its solution $y = y_{(m)}(N, \tau; t)$ for $t \in \omega_\tau$ exists for arbitrary φ_α and $y(0)$ from H_N and depends uniformly with respect to N, τ continuously on φ_α and $y(0)$, $\alpha = 1, 2, \dots, m$ (see (8)), so that, for example,

$$\max_{\omega_\tau} \|y(t)\|_N \leq M_1 \|y(0)\|_N + M_2 \max_{\alpha, \omega_\tau} \|\varphi_\alpha(t)\|_N. \quad (6)$$

Here and below M_k ($k = 1, 2, \dots$) are positive constants independent of N and τ . For correctness of the L.o.s. (5) it is sufficient that the inverse operator $C_\alpha = (R_\alpha^N)^{-1}$ exist and that the conditions

$$\begin{aligned} \|B_m(t) \dots B_1(t)\|_N &\leq 1 + M_3 \tau, \quad \|B_m(t) \dots B_{s+1}(t) C_s(t)\| \leq M_4, \\ \|C_m(t)\|_N &\leq M_5, \quad t \in \omega_\tau, \end{aligned} \quad (7)$$

be satisfied, where $B_\alpha = (R_\alpha^N)^{-1} S_\alpha^N$; $s = 1, 2, \dots, m-1$.

Theorem 2. *Let the conditions of Theorem 1 be satisfied, let the L.o.s. (5) be correct, and let each of the one-dimensional schemes (5_α) of number α approximate the corresponding equation (3_α) on its solution $v_{(\alpha)}(t)$. Then the solution of problem (5) converges uniformly in t to the solution of problem (1) as $\tau \rightarrow 0$ and $N \rightarrow \infty$:*

$$\lim_{N \rightarrow \infty, \tau \rightarrow 0} \|y - P_N u\|_N = 0$$

for $t \in \omega_\tau$.

We do not dwell here on the question of the order of accuracy of the L.o.s. (see 5-7,2). We note that, by passing successively from (1) to (3), (4), and (5), one can strengthen some estimates (5-7). If the condition $\tilde{U} = U$ is fulfilled and, consequently, $v^j = u^j$, then for the solution of the one-dimensional problems (3_α) it is expedient to use schemes of order of accuracy higher than $O(\tau)$, since $y^j - P_N u^j = y^j - P_N v^j$ (for example, a scheme $O(h^4 + \tau^2)$ for the heat-conduction equation with allowance for the method of computing $f_m(t)$ indicated above, etc.).

7. Let H_N be a Euclidean space with scalar product $(y, z)_N$ and norm $\|z\|_N = \sqrt{(z, z)_N}$. Consider the family of two-level L.o.s. of the form

$$\begin{aligned} \frac{y_{(\alpha)} - y_{(\alpha-1)}}{\tau} + A_\alpha^N (\sigma_\alpha y_{(\alpha)} + (1 - \sigma_\alpha) y_{(\alpha-1)}) &= \varphi_\alpha, \\ \alpha = 1, \dots, m, \quad t \in \omega_\tau, \quad y(0) &= P_N u_0, \end{aligned} \quad (8)$$

where A_α^N is a linear operator whose domain of definition and range coincide with H_N . For correctness of the L.o.s. (8) one of three groups of conditions is sufficient for $t \in \omega_\tau$ (cf. (8)):

Theorem 3. If $\sigma_\alpha \geq 0.5$, $A_\alpha^N(t)$ is semibounded from below, i.e.

$$(A_\alpha^N(t)y, y)_N \geq -c_1 \|y\|_N^2$$

for any $y \in H_N$, and $\tau \leq \tau_0(c_1)$ is sufficiently small, then the L.o.s. is correct.

Theorem 4. If $\sigma_\alpha \geq 0.5$, $(A_\alpha^N y, y)_N \geq 0$, $y \in H_N$, $\tau > 0$ is arbitrary, then the L.o.s. is correct.

Theorem 5. If $\sigma_\alpha \geq \sigma_\varepsilon = 0.5 - (1 - \varepsilon)/\tau \|A_\alpha^N\|_N$, $(A_\alpha^N y, y)_N \geq 0$, A_α^N is a finite-dimensional self-adjoint operator, i.e. $(A_\alpha^N y, v)_N = (y, A_\alpha^N v)_N$, $\varepsilon \in (0, 1]$, then the l.o.s. is correct.

8. Changing f_α , the numbering of A_α in (2), the number of terms in (2), etc., we obtain an innumerable set of systems (3)–(4). In particular, putting

$$A = \sum_{\alpha=1}^m A_\alpha = \sum_{\alpha=1}^m A'_\alpha, \quad A'_\alpha = \frac{1}{2} A_\alpha, \quad \alpha \leq m, \quad A'_\alpha = \frac{1}{2} A_{2m+1-\alpha}, \quad \alpha > m,$$

we obtain the “symmetric” system (3), (4), which in a number of cases has accuracy $O(\tau^2)$ ($\|v^j - u^j\| = O(\tau^2)$). For $m = 2$ we have $A'_1 = \frac{1}{2} A_1$, $A'_2 = \frac{1}{2} A_2$, $A'_3 = \frac{1}{2} A_2$, $A'_4 = \frac{1}{2} A_1$ (the scheme $\frac{1}{2} A_1 \rightarrow \frac{1}{2} A_2 \rightarrow \frac{1}{2} A_2 \rightarrow \frac{1}{2} A_1$). Choosing in (8) $\sigma_1 = 0$, $\sigma_2 = 1$, $\sigma_3 = 0$, $\sigma_4 = 1$, we obtain a generalization of scheme (1), as may be verified after eliminating $y_{(1)}$, $y_{(3)}$. (This was also pointed out by I. V. Fryazinov.) Thus, scheme (1) is an l.o.s. One may also use an l.o.s. with $\sigma_1 = \sigma_2 = 0$, $\sigma_3 = \sigma_4 = 1$. The indicated symmetrization makes it possible to obtain second-order accuracy in τ .

It is possible to use three-layer schemes (for $m = 2$ this is obvious), and also combinations of two-layer and three-layer schemes. Schemes (3) for the homogeneous ($f = 0$) heat equation are more naturally interpreted as l.o.s., and not as splitting schemes.

9. In (5) a somewhat different method of one-dimensional modeling of problem (1) was proposed. The system considered was

$$\frac{1}{m} \frac{dv_{(\alpha)}}{dt} + A_\alpha(t) v_{(\alpha)} = f_\alpha(t), \quad t \in (t_{\alpha-1}^j, t_\alpha^j), \quad t_\alpha^j = t^j + \frac{\alpha}{m} \tau, \quad \alpha = 1, 2, \dots, m,$$

$$v_{(\alpha)}(t_{\alpha-1}^j) = v_{(\alpha-1)}(t_{\alpha-1}^j), \quad \alpha > 1, \quad v_{(1)}(t^j) = v(t^j) = v_{(m)}(t^j),$$

where $v(t^{j+1}) = v_{(m)}(t^{j+1})$. If $f_\alpha = 0$ and A_α do not depend on t , then this problem is equivalent to problem (3), (4). In the general case their solutions differ by $O(\tau)$. An analogue of Theorem 1 holds (see also (10)).

Hence the same l.o.s. as before are obtained (see (9)).

10. The case

$$A = \sum_{\alpha, \beta=1}^p A_{\alpha\beta}$$

has also been studied. System (3) has the form ($m = 2p$)

$$\frac{dv_{(\alpha)}}{dt} + \sum_{\beta=1}^{\alpha} A_{\alpha\beta}^{-}(t)v_{(\beta)} = f_{\alpha}, \quad \alpha \leq p; \quad \frac{dv_{(\alpha)}}{dt} + \sum_{\beta=p}^{2p+1-\alpha} A_{2p+1-\alpha, \beta}^{+}(t)v_{(\beta)} = f_{\alpha},$$

$$\alpha > p,$$

with the initial conditions (4); here $(A_{\alpha\beta}^{-})$ and $A_{\alpha\beta}^{+}$ are triangular matrix-operators ($A_{\alpha\beta}^{-} = 0$ for $\beta > \alpha$, $A_{\alpha\beta}^{+} = 0$ for $\beta < \alpha$, $A_{\alpha\beta}^{-} + A_{\alpha\beta}^{+} = A_{\alpha\beta}$). The passage to an l.o.s. is evident. Such a method for constructing l.o.s. was used in (6) for a system of multidimensional parabolic equations with mixed derivatives.

11. Locally one-dimensional schemes were studied for quasilinear equations, and also for equations of second order (see (7)):

$$\frac{d^2 u}{dt^2} + A(t)u = f(t), \quad u(0) = u_0, \quad \frac{du}{dt}(0) = \bar{u}_0, \quad A = \sum_{\alpha=1}^m A_{\alpha}, \quad A = \sum_{\alpha=1}^m (A_{\alpha} + A_{\alpha}^{*}),$$

where A_{α}^{*} is the operator adjoint to A_{α} .

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