

ON THE CONVERGENCE OF LACUNARY TRIGONOMETRIC SERIES ON SETS

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1965

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Abstract

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UDC 517.512.2

MATHEMATICS

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ON THE CONVERGENCE OF LACUNARY TRIGONOMETRIC SERIES ON SETS

(Presented by Academician A. N. Kolmogorov, 15 IV 1965)

Zygmund proved the following theorems:

Theorem 1. If the upper limit of indeterminacy of a lacunary trigonometric series is different from $+\infty$ at each point of some interval, then the series of the moduli of the coefficients converges.

Theorem 2. If a lacunary trigonometric series is summable on a set of positive measure by a method generated by a matrix in which each column forms a sequence converging to zero, and the sums of the numbers written in the rows form a sequence converging to one, then the series of the squares of the coefficients converges.

A. N. Kolmogorov proved that, if a Fourier series turns out to be lacunary, then it converges almost everywhere. In the present note we indicate sets by which one can replace the interval in Theorem 1 and the set of positive measure in Theorem 2 without violating the validity of the assertions made in them. S. B. Stechkin proved that the interval in Theorem 1 can be replaced by a set of the second category. Therefore we shall be interested only in sets of the first category and, first of all, perfect sets of measure zero.

Definition 1. Let $\omega(n) \downarrow 0$ as $n \rightarrow \infty$. A set $A \subseteq [0, 2\pi]$ is called an $M\text{-}\omega(n)$ -set if there exists on $[0, 2\pi]$ a nondecreasing function $F(x)$, constant on every interval (α, β) contained in $[0, 2\pi]$ and not containing points of A , not identically constant on $[0, 2\pi]$, and such that

$$\int_0^{2\pi} e^{-inx} dF(x) = O(\omega(n)).$$

The interval $[\alpha, \beta] \subseteq [0, 2\pi]$ is an $M\text{-}n^{-1}$ -set. D. E. Men' shov was the first to construct a perfect set of measure zero which is an $M\text{-}\ln^{-1/2} n$ -set. Littlewood was the first to prove the existence, for $\varepsilon > 0$, of a set of measure zero which is an $M\text{-}n^{-\varepsilon}$ -set. Examples of perfect sets of measure zero which are $M\text{-}n^{-\varepsilon}$

$(0 < \varepsilon < 1/2)$ -sets can be found, for example, in Salem's work ⁽⁵⁾. In the present work the following is used essentially.

Lemma 1. Let on $[N, +\infty]$ a function $\omega(t)$ be given, having, at $+\infty$, the following properties: 1) $\omega(t) \downarrow 0$; 2) $\omega^{-1}(t) = O(t^2)$; 3) for any constant $C > 0$ there exists a constant C_1 such that $\omega(Ct) \leq C_1\omega(t)$. If the upper limit of a sequence of functions continuous on $[0, 2\pi]$ is different from $+\infty$ on some M - $\omega(n)$ -set, then there exists an M - $\omega(n)$ -set on which the sequence is uniformly bounded above.

Theorem 3. Let $\omega(t)$ satisfy the conditions of Lemma 1, $n_k \uparrow \infty$, $\Delta n_k = n_k - n_{k-1} \uparrow \infty$,

$$\sum_{k=1}^{\infty} k\omega^2(\Delta n_k) < +\infty.$$

If

$$\overline{\lim}_{N \rightarrow \infty} \left| \sum_{k=1}^N \rho_k \cos(n_k x + \varepsilon_k) \right| \neq \infty$$

on an $M - \omega(n)$ -set, then

$$\sum_{k=1}^{\infty} \rho_k^2 < \infty.$$

Theorem 4. Let $\omega(t)$ satisfy the conditions of Lemma 1, let the sequence $\{n_k\}$ satisfy the conditions of Theorem 3, and let the matrix $A = (a_{nk})$ be such that: 1) for every n there exists k_n such that $a_{nk} = 0$ for $k > k_n$; 2) for every fixed k ,

$$\lim_{n \rightarrow \infty} a_{nk} = 0;$$

3)

$$\lim \sum_{k=1}^{k_n} a_{nk} = 1.$$

If

$$\sum_{k=1}^{\infty} \rho_k \cos(n_k x + \varepsilon_k)$$

is summable to a finite number at each point of an $M - \omega(n)$ -set by the method generated by the matrix A , then

$$\sum_{k=1}^{\infty} \rho_k^2 < +\infty.$$

From Theorems 3 and 4, as a corollary, we obtain the following theorems.

Theorem 5. If the limits of indeterminacy of a lacunary trigonometric series are finite at each point of some $M - \ln^{-1-\delta} n$ -($\delta > 0$) set, then it is a Fourier series.

Theorem 6. Let the matrix A satisfy the conditions of Theorem 4. If a lacunary trigonometric series is summable at each point of some $M - \ln^{-1-\delta} n$ -($\delta > 0$) set by the method generated by the matrix A , then it is a Fourier series.

The following theorem is a generalization of Theorem 1.

Theorem 7. If the upper limit of indeterminacy of a lacunary trigonometric series is different from $+\infty$ on some $M - n^{-\delta}$ -($\delta > 0$) set, then it is the Fourier series of a continuous function.

Definition 2. A set E is called a $Z[\lambda]$ -set if, from the finiteness of the upper limit of indeterminacy on E of a lacunary trigonometric series with degree of lacunarity greater than λ , there follows the convergence of the series of the moduli of its coefficients.

Definition 3. A set E is called a Z -set if there exists λ for which E is a $Z[\lambda]$ -set.

In the work of V. V. Nemytskii ⁽⁴⁾, the concept of the A - k -property for closed sets was introduced.

Theorem 8. If a set E has the A - k -property and $k > 1$, then it is a Z -set.

It follows from Theorem 8 that there exist sets of uniqueness for the trigonometric series which are Z -sets. One can construct sets of uniqueness for the trigonometric series which are $Z[\lambda]$ -sets for any $\lambda > 5$. But it is not known whether there exist sets of uniqueness for the trigonometric series which are $Z[1]$ -sets.

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Received
15 IV 1965

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Note: Figure translations are in progress. See original paper for figures.

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