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V. A. Suchkov

1965

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Abstract

Full Text

V. A. SUCHKOV

PLANE POTENTIAL FLOWS WITH STATIONARY STREAMLINES

(Presented by Academician M. A. Lavrent'ev on February 20, 1965)

Hydromechanics

§ 1. Consider the equations of plane potential motion of a polytropic gas

$$\frac{\partial u_i}{\partial t} + \frac{\partial \Delta}{\partial x_i} = 0, \quad \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1}; \quad (1,1)$$

$$\frac{\partial \Delta}{\partial t} + 2u_k \frac{\partial \Delta}{\partial x_k} - u_i u_k \frac{\partial u_i}{\partial x_k} + c^2 \frac{\partial u_k}{\partial x_k} = 0; \quad (1,2)$$

$$c^2 = \chi \left(\Delta - \frac{u_k^2}{2} \right), \quad i, k = 1, 2. \quad (1,3)$$

Here x_i are Cartesian coordinates, t is time, u_i are the components of the velocity vector, c is the speed of sound, and Δ is the total energy per unit mass. For an adiabatic flow $\chi = \gamma - 1$; for an isothermal flow $\chi = 0$, $\chi\Delta = 1$.

Plane flows with stationary streamlines are characterized by the differential relation

$$u_2 \frac{\partial u_1}{\partial t} - u_1 \frac{\partial u_2}{\partial t} = 0. \quad (1,4)$$

It is necessary to investigate the compatibility of the overdetermined system (1,1)–(1,4) (see ⁽¹⁾). This class of flows includes stationary flows and nonstationary flows with plane or axial symmetry, where the stationary streamlines are rectilinear ^(2,3).

§ 2. Take as new unknown functions the velocity modulus v and θ , the angle of inclination of the velocity vector to the x_1 -axis; then

$$u_1 = v \cos \theta, \quad u_2 = v \sin \theta. \quad (2,1)$$

Substituting (2,1) into (1,4), we obtain

$$\frac{\partial \theta}{\partial t} = 0. \quad (2,2)$$

Let us write the equations of potential flows with stationary streamlines in curvilinear orthogonal coordinates q_1, q_2 ($q_2 = \text{const}$ are the streamlines, $q_1 = \text{const}$ are the lines orthogonal to them):

$$\frac{\partial \psi}{\partial t} + \frac{\partial \Delta}{\partial q_1} = 0, \quad \frac{\partial \psi}{\partial q_2} = 0, \quad \frac{\partial \Delta}{\partial q_2} = 0; \quad (2,3)$$

$$\frac{\partial \Delta}{\partial t} + 2\psi \frac{\partial \Delta}{\partial q_1} l - \psi^2 \frac{\partial \psi}{\partial q_1} l^2 - \psi^3 m + c^2 \left(l \frac{\partial \psi}{\partial q_1} + \psi n \right) = 0; \quad (2,4)$$

$$\frac{\partial}{\partial q_1} \left(\frac{1}{H_1} \frac{\partial H_2}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{1}{H_2} \frac{\partial H_1}{\partial q_2} \right) = 0, \quad (2,5)$$

where

$$\psi = vH_1, \quad l = \frac{1}{H_1^2}, \quad m = \frac{l}{2} \frac{\partial l}{\partial q_1}, \quad (2,6)$$

$$n = \frac{1}{2} \frac{\partial l}{\partial q_1} + \frac{l}{H_2} \frac{\partial H_2}{\partial q_1}, \quad c^2 = \chi \left(\Delta - \frac{\psi^2}{2} l \right).$$

§ 3. Suppose that the Lamé coefficients are known and satisfy equation (2,5) ⁽⁴⁾. Consider the compatibility of the system (2,3), (2,4). The func-

χ , Δ , and ψ depend only on q_1, t , while the functions l, m, n depend on q_1, q_2 . The variable q_2 enters only into equation (2,4) as a parameter. Differentiating this equation with respect to q_2 and setting the differential consequences equal to zero, we obtain compatibility conditions for the functions $H_1(q_1, q_2)$, $H_2(q_1, q_2)$.

Theorem 1. *A plane unsteady potential flow with rectilinear stationary streamlines is a motion with plane or axial symmetry.*

Theorem 2. *A plane flow with curvilinear stationary streamlines, possessing a functional Jacobian, is either stationary or belongs to the class of solutions:*

1. $\chi = 0$, $\psi = 1$, $\Delta = \alpha t + \beta$, α, β are constants;
2. $\chi \neq 0$, $\psi = t^{-1}$, $\Delta = q_1 t^{-2}$.

The functions H_1, H_2 are determined from a system of two differential equations (2,5) and, respectively, (3.1) or (3.2):

$$\alpha + n = m; \quad (3.1)$$

$$\chi n(q_1 - l/2) - m - 2(q_1 - l) = 0. \quad (3.2)$$

Theorem 3. *The functions $\psi = t^{-1}$, $\Delta = q_1 t^{-2} + s$, where for $\chi \neq 2$, $s = s_0 t^{-4\chi/(\chi+2)}$; for $\chi = 2$, $s = -t^{-2} \ln t$; s_0 is constant, describe a plane flow with curvilinear streamlines having a constant Jacobian. The Lamé coefficients H_1, H_2 are known functions of q_1, q_2 , for example, for $\chi = 2$*

$$H_1^2 + H_2^2 = 1, \quad H_1^{-2} + \ln(H_1^{-2} - 1) = 2(q_1 + q_2). \quad (3.3)$$

§ 4. From the condition of stationarity of the streamlines we have obtained an explicit dependence of the solutions on time in the variables q_1, q_2, t . This dependence is unchanged also in the variables x_1, x_2, t , so that the solution must have the form

$$u_i = \psi V_i, \quad \Delta = -\psi' V + s, \quad (4.1)$$

where ψ, s are known functions of t ; V and $V_i = \partial V / \partial x_i$ are functions only of x_1, x_2 .

For $\chi = 0$, $\psi = 1$, $s = \alpha t + \beta$, for the function $V(x_1, x_2)$ from (1.2) we obtain the equation

$$(V_i V_k - \delta_{ik}) \cdot V_{ik} = \alpha. \quad (4.2)$$

The function $V(x_1, x_2)$ for $\chi \neq 0$, $\psi = t^{-1}$, $s = 0$ must satisfy the equation

$$[V_i V_k - \chi(V - V_i^2/2)\delta_{ik}] V_{ik} - 2V_i^2 + 2V = 0. \quad (4.3)$$

The Jacobian of the solution of equations (4.2) and (4.3) is two functions of one argument.

For $\chi = 2$, $\psi = t^{-1}$, $s = -t^{-2} \ln t$, integrating the system

$$\begin{aligned} \partial x_1 / \partial q_1 &= H_1 \cos \theta, & \partial x_1 / \partial q_2 &= -H_2 \sin \theta, \\ \partial x_2 / \partial q_1 &= H_1 \sin \theta, & \partial x_2 / \partial q_2 &= H_2 \cos \theta, \end{aligned} \quad (4.4)$$

we obtain the solution

$$q_1 = x_1 + \frac{1}{2}(x_2^2 + 1) = V(x_1, x_2). \quad (4.5)$$

Similarly, for $\chi \neq 2$, $\psi = t^{-1}$, $s = s_0 t^{-4\chi/(\chi+2)}$ we find

$$q_1 = \frac{1}{2} \left(x_2^2 + \frac{2-\chi}{2+\chi} x_1^2 \right) = V(x_1, x_2). \quad (4.6)$$

I take this opportunity to express my deep gratitude to N. N. Yanenko for his constant attention and interest in the work.

Received 30 XI 1964

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