

ON QUASIANALYTIC FUNCTIONS IN $\backslash(L(-\infty, \infty)\backslash)$

S. P. Geisberg

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Abstract

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MATHEMATICS

S. P. GEISBERG

ON QUASIANALYTIC FUNCTIONS IN $L(-\infty, \infty)$

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1. Usually, in order to obtain necessary and sufficient conditions for the quasianalyticity of a function, one has to consider quasianalytic classes of functions ^(1,2); this is caused by the difficulties that arise in proving that the corresponding sufficient conditions are also necessary. Thus, for example, according to Mandelbrojt ⁽²⁾, the class $L(M_n)$ of functions $f(x)$ infinitely differentiable in the space* and defined by the conditions $\|f^{(n)}\|_L \leq M_n$, is called quasianalytic if, for every function $f \in L(M_n)$, the closures of the linear spans of its derivatives $f^{(n)}$ and of its translates $f_x(t) = f(x+t)$ coincide. The question naturally arises of such a definition of the concept of a quasianalytic function which would make it possible to give necessary and sufficient conditions of quasianalyticity referring to a function, and not to a class of functions. In this note it is shown that for L one can propose the following definition.

Definition. A function $f \in L$ is called *A-quasianalytic* if, for every functional $m \in M$ and every $\{x_k\}$, $x_k \neq x_0$, $x_k \rightarrow x_0$, from the conditions

$$m(f_{x_k}) = \int_{-\infty}^{\infty} f(x_k + t)m(t) dt = 0, \quad k = 1, 2, \dots, \infty,$$

it follows that $m(f_x) = 0$.

Obviously, every function belonging to a quasianalytic class of Mandelbrojt is *A-quasianalytic*. The properties of *A-quasianalytic* functions are established in the following theorems.

Theorem 1. *If f is an *A-quasianalytic* function, then f is infinitely differentiable, $f^{(n)} \in L(-\infty, \infty)$, and** *

$$\lim_{\tau \rightarrow 0} \left\| \left(f_\tau - \sum_{j=0}^n \frac{\tau^j}{j!} f^{(j)} \right) \tau^{-(n+1)} - \frac{f^{(n+1)}}{(n+1)!} \right\|_L = 0,$$

$$f^{(0)} = f, \quad n = 0, 1, \dots, \infty. \quad (1)$$

Theorem 2. Let $f(t)$ be an even function, $f \in L$, and let its Fourier transform $F(x)$ satisfy the conditions:

- 1°. $F(x) > 0$, is differentiable, and for $x > 0$, $-xF'(x)/F(x)$ strictly increases.
- 2°. For some $\beta > 0$, $|F'(x)| = O(|x|^\beta + 1)$.

Then, in order that $f(t)$ be an A -quasianalytic function, it is necessary and sufficient that the following conditions hold:

- 3°. $f(t)$ is infinitely differentiable, $f^{(n)} \in L$, $n = 1, 2, \dots, \infty$, and (1) holds.
- 4°.

$$\int_1^\infty \frac{\log F(x)}{x^2} dx = -\infty.$$

* In what follows, L and M denote the real space $L(-\infty, \infty)$ and its conjugate.

** Formula (1) should be regarded as a recurrence relation defining the successive derivatives of the function f .

Conditions 1°, 2° occur in (2). Theorem 2, together with (2), p. 52, Lemma 1; p. 38, Theorem 1, shows that for functions satisfying conditions 1°, 2°, A -quasianalyticity and quasianalyticity in the sense of Mandelbrojt coincide. Theorems 1 and 2 may also be regarded as a development of Wiener's theorem (3), since they give conditions for closedness in L of any system of translates f_x having a limit point $x_k \rightarrow x_0$.

2. Proof of Theorem 1. We restrict ourselves to the case $n = 1$. For $\tau > 0$ put $\varphi_\tau = (f_\tau - f)/\tau$, and first show that $\|\varphi_\tau\|_L = O(1)$. Indeed, otherwise it is not difficult to show that $\psi_\tau = \varphi_\tau/\|\varphi_\tau\|_L$ tends weakly to zero as $\tau \rightarrow 0+$. Then, by induction, one can construct a sequence of functionals $m_{\tau_k} \in M$ such that

$$m_{\tau_0}(\psi_{\tau_0}) = 1; \quad m_{\tau_k}(f) = 0, \quad k = 0, 1, \dots, \infty; \quad m_{\tau_k}(\psi_{\tau_j}) = 0, \quad 0 \leq j < k;$$

$$m_{\tau_k}(\psi_{\tau_k}) = -\sum_{j=0}^{k-1} m_{\tau_j}(\psi_{\tau_k}); \quad \|m_{\tau_k}\|_M < 2^{-k},$$

where $\{\psi_{\tau_k}\}$ is a suitably chosen subsequence of $\{\psi_\tau\}$, $\tau_k \downarrow 0$. Consequently, for the functional $m = \sum m_j$ we have $m \in M$ and $m(f_{\tau_0}) = m(\psi_{\tau_0}) = 1$, $m(f_{\tau_k}) = m(\psi_{\tau_k}) = 0$, $k = 1, 2, \dots, \infty$, which contradicts the A -quasianalyticity of the function f .

Next, without loss of generality, one may suppose that for some σ , $F(x) \neq 0$ for $|x| \leq \sigma$. Denote by $\sigma_{\alpha, \beta}(z)$ the entire function of degree σ for which $\sigma_{\alpha, \beta}(0) = \alpha$,

$\sigma'_{\alpha,\beta}(0) = \beta$, and $\|\sigma_{\alpha,\beta}(x)\|_L < \infty$. From the Wiener-Paley theorems ⁽⁴⁾ and Wiener ⁽³⁾ it follows that there exists $m_{\alpha,\beta} \in M$ such that $\sigma_{\alpha,\beta}(x) \equiv m_{\alpha,\beta}(f_x)$. We show that this implies the weak convergence of $\{\varphi_\tau\}$ as $\tau \downarrow 0$. In fact, since $\|\varphi_\tau\|_L = o(1)$, otherwise there would exist a functional $m \in M$, subsequences $\{\tau'_j\}$, $\{\tau''_j\}$, and numbers $a < b$ such that $\tau'_j \downarrow 0$,

$$\tau''_j \downarrow 0, \quad m(\varphi_{\tau'_j}) \rightarrow a, \quad m(\varphi_{\tau''_j}) \rightarrow b.$$

Set $m(f) = \alpha$, $(a + b)/2 = \beta$, $\bar{m} = m - m_{\alpha,\beta}$. Then $\bar{m} \in M$ and

$$\begin{aligned} \bar{m}(f_{\tau'_j}) &= m(f) + \tau'_j \bar{m}(\varphi_{\tau'_j}) = \frac{1}{2}(a - b)\tau'_j + o(\tau'_j); \quad \bar{m}(f_{\tau''_j}) = m(f) + \tau''_j \bar{m}(\varphi_{\tau''_j}) \\ &= \frac{1}{2}(b - a)\tau''_j + o(\tau''_j). \end{aligned}$$

This means that the continuous function $\bar{m}(f_x)$ changes sign in every neighborhood to the right of zero. Consequently, the point $x = 0$ is a limit point of its zeros. Since, obviously, $\bar{m}(f_x) \not\equiv 0$, this contradicts the A -quasianalyticity of the function f .

By the weak completeness of L , the sequence $\{\varphi_\tau\}$ as $\tau \downarrow 0$ has a weak limit $\varphi_{0+} \in L$. Considering the functions $\varphi_\tau - \varphi_{0+}$, which converge weakly to 0, and using arguments analogous to those given above, we obtain that, as $\tau \downarrow 0$, $\|\varphi_\tau - \varphi_{0+}\|_L \rightarrow 0$. In exactly the same way one establishes the existence of a function $\varphi_{0-} \in L$ for which $\|\varphi_\tau - \varphi_{0-}\|_L \rightarrow 0$ as $\tau \uparrow 0$. Obviously, the Fourier transforms of the functions φ_{0+} and φ_{0-} coincide. Consequently, φ_{0+} and φ_{0-} coincide almost everywhere. Therefore

$$\lim_{\tau \rightarrow 0} \|\varphi_\tau - \varphi_{0+}\|_L = \lim_{\tau \rightarrow 0} \|(f_\tau - f)/\tau - \varphi_{0+}\|_L = 0, \quad \text{i.e. } f' = \varphi_{0+}.$$

3. Proof of Theorem 2. First of all, conditions 1°–4° are sufficient, since from 1°–4° and ⁽²⁾, p. 52, Lemma 1; p. 38, Theorem 1, it follows that for every $m \in M$ the function $\mu(x) = m(f_x)$ belongs to the quasianalytic class of functions (1); consequently, its zeros have no limit point, i.e. f is an A -quasianalytic function.

The necessity of condition 3° follows from Theorem 1. Let us prove the necessity of 4°. Suppose

$$\int_1^\infty \frac{\log F(x)}{x^2} dx > -\infty.$$

Then, using 1°, one can show that if $\nu > 0$, $x(\nu) = -F'(\nu)\nu/F(\nu) > 0$, then

$$x(\nu) \uparrow \infty, \quad \min_{\nu \leq t \leq \nu+1/2} F(t)t^{x(\nu)} \geq \frac{1}{2}F(\nu)\nu^{x(\nu)}. \quad (2)$$

Moreover, if $x \geq 1$,

$$v(x) = \max_{r \geq x} r^{x - [\ln x]} F(r), \quad c > 1, \quad \psi_c(r) = \max_{r \geq x \geq 1} \frac{(cr)^x}{v(x)},$$

then

$$\int_1^\infty \frac{\ln \psi_c(r)}{r^2} dr < \infty. \quad (3)$$

Put $v(x) = 1$, $0 \leq x < 1$, and choose λ_n so that $\{\lambda_n\}$ is increasing and the λ_n run through all values from the set of numbers $2, 4, 6, \dots, 2n, \dots; 1^2, 3^2, 5^2, \dots, (2n+1)^2, \dots$. Then, for $r \geq 1$ and some $D > 0$,

$$\theta(r) = 2 \sum_{\lambda_n \leq r} \frac{1}{\lambda_n} \leq \ln Dr, \quad U(r) = e^{\theta(r)} \leq Dr,$$

$$V(r) = \sup_{0 \leq x \leq r} \frac{[U(r)]^x}{v(x)} = O(\psi_D(r)),$$

and, by virtue of (3),

$$\int_1^\infty \frac{\ln V(r)}{r^2} dr < \infty. \quad (4)$$

Next, let $V(r) = V(-r)$ and $V(r) = 1$ for $|r| < 1$. Put

$$g(z) = \frac{2x}{\pi} \int_{-\infty}^\infty \frac{\ln V(\tau) d\tau}{x^2 + (y - \tau)^2}, \quad x > 0, \quad z = x + iy;$$

(z) is the function conjugate to $g(z)$; $h(x) \equiv 0$ for $x > 0$;

$$\omega(z) = e^{-g(z) - ih(z)};$$

$$G(z) = \prod_{n=1}^\infty \frac{\lambda_n - z}{\lambda_n + z} e^{2z/\lambda_n}; \quad \Phi(z) = C^{-(1+z)}(1+z)^{-A} G(z) \omega(1+z),$$

where A and C are sufficiently large positive numbers. From (4) and (5), p. 19, it follows that, for $x \geq 0$, $\Phi(z)$ has the representation

$$\Phi(z) = \int_0^\infty \varphi(u) u^z du,$$

where

$$\int_0^\infty |\varphi(u)| u^x du = O(v(x)). \quad (5)$$

In particular, setting $\varphi(-u) = \varphi(u)$, we have

$$\operatorname{sgn} \int_{-\infty}^\infty |\varphi(u)| u^x du = \operatorname{sgn} G(x). \quad (6)$$

Let us show that $u\varphi(u)/F(u) \in L$. Indeed, the function $r^{xF}(r)$ attains its maximum at r satisfying the equality $x = x(r)^*$. Therefore, setting $x_n = x(n/4)$, $n = 1, 2, \dots, \infty$, we have

$$v(x_n+3) = \max_{r \geq x_n+3} r^{x_n+3-[\ln(x_n+3)]} F(r) = O(1) \max_{r>0} r^{x_n} F(r) = O(1)(n/4)^{x_n} F(n/4).$$

* Recall that $x(\nu) = -F'(\nu)\nu/F(\nu)$ and $x'(\nu) \uparrow \infty$.

Moreover, for $n/4 \leq u \leq (n+1)/4$, by (2), $u^{x_n} F(u) \geq \frac{1}{2}(n/4)^{x_n} F(n/4)$. Hence, by (5),

$$\begin{aligned} \int_{n/4}^{(n+1)/4} u \frac{|\varphi(u)|}{F(u)} du &= \int_{n/4}^{(n+1)/4} u^{-2} \frac{|\varphi(u)| u^{3+x_n}}{F(u) u^{x_n}} du \leq \\ &\leq \left(\frac{n}{4}\right)^{-2} \frac{2}{(n/4)^{x_n} F(n/4)} \int_0^\infty |\varphi(u)| u^{x_n+3} du = O(1) \frac{\nu(x_n+3)}{(n/4)^{x_n} F(n/4)} = O(n^{-2}). \end{aligned}$$

Thus,

$$\int_{-\infty}^\infty |u| \frac{|\varphi(u)|}{F(u)} du = O(1) \left[\sum_1^\infty n^{-2} + \int_0^1 |u\varphi(u)| du \right] < \infty.$$

Next, put

$$m(x) = \int_{-\infty}^\infty \frac{u\varphi(u)}{F(u)} \sin ux du.$$

Then $m(x) \in M$ and

$$m(fx) = \int_{-\infty}^{\infty} u\varphi(u) \sin ux \, du = \Psi(x).$$

We shall show that $x = 0$ is a limit point of the zeros of $\Psi(x)$, and thereby prove the theorem*. For $2n < x < 2(n+1)$ we have

$$\begin{aligned} & \int_0^{\infty} \left(\int_{-\infty}^{\infty} \varphi(u)(1 - \cos u\tau) \, du \right) \tau^{-(x+1)} \, d\tau = \\ &= \int_0^{\infty} \left(\int_{-\infty}^{\infty} \varphi(u) \left(\sum_{k=0}^n (-1)^k \frac{\tau^{2k} u^{2k}}{(2k)!} - \cos u\tau \right) \, du \right) \tau^{-(x+1)} \, d\tau = \\ &= \int_{-\infty}^{\infty} \varphi(u) \, du \int_0^{\infty} \left(\sum_{k=0}^n (-1)^k \frac{\tau^{2k} u^{2k}}{(2k)!} - \cos u\tau \right) \tau^{-(x+1)} \, d\tau = \\ &= (-1)^n \frac{\Gamma(x+1)}{\Gamma(x+1-2n)} \int_0^{\infty} \frac{1 - \cos u\tau}{\tau^{x+1-2n}} \, d\tau \int_{-\infty}^{\infty} \varphi(u) |u|^x \, du, \end{aligned}$$

since, by (6),

$$\int_{-\infty}^{\infty} \varphi(u) u^{2k} \, du = 0, \quad k = 1, 2, \dots, \infty.$$

Consequently, if

$$\omega(\tau) = \int_{-\infty}^{\infty} \varphi(u) \cos u\tau \, du,$$

then, by (6), for $2n < x < 2n+1$,

$$\operatorname{sgn} \int_0^{\infty} \frac{\omega(0) - \omega(\tau)}{\tau^{x+1}} \, d\tau = (-1)^n \operatorname{sgn} G(x).$$

It follows from this and from the definition of $G(x)$ that on each interval

$$(2n+1)^2 - 1 < x < (2n+1)^2 + 1$$

the function

$$R(x) = \int_0^\infty \frac{\omega(0) - \omega(\tau)}{\tau^{x+1}} d\tau$$

changes sign. Since $\omega(\tau) = O(1)$, this is possible only if $\omega(\tau) \equiv 0$ in some right-hand neighborhood of 0, or if $\omega(\tau)$ changes sign in every right-hand neighborhood of 0. In both cases $\tau = 0$ must be a limit point of the zeros of $\omega'(\tau)$. But $\omega'(\tau) = \Psi(\tau)$; therefore $\tau = 0$ is a limit point of the zeros of $\Psi(\tau)$.

Leningrad Civil Engineering Institute

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* Obviously, $\Psi(x) \not\equiv 0$.

Note: Figure translations are in progress. See original paper for figures.

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