
Soviet-era science, translated into English

Reports of the Academy of Sciences of the USSR

1965

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196501.23479>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1965. Vol. 163, No. 5

MATHEMATICS

V. I. GURARII

SPACES OF UNIVERSAL DISPOSITION

(Presented by Academician L. V. Kantorovich on 25 I 1965)

Let E be a Banach space; P a subspace in E ; $\mathfrak{M}$ a certain class of Banach spaces; $\mathfrak{R}$ the class of all finite-dimensional Banach spaces; $\mathfrak{H}$ the class of all separable Banach spaces.

Definition 1. We shall call E a -universal ($a \geq 1$) with respect to $\mathfrak{M}$ if for every $F \in \mathfrak{M}$ there is a subspace $\tilde{F}$ in E and an isomorphism φ of F onto $\tilde{F}$ such that

$$\max\{\|\varphi\|, \|\varphi^{-1}\|\} \leq a.$$

If E is a -universal with respect to $\mathfrak{M}$ for every $a > 1$ (respectively for $a = 1$), then we shall call E almost universal (respectively universal) with respect to $\mathfrak{M}$.

Thus, for example, each of the spaces C, m, A^* is universal with respect to $\mathfrak{H}$ (^{1,2}); one can show that c is almost universal with respect to $\mathfrak{R}$.

Definition 2. We shall call P a subspace of a -universal ($a \geq 1$) disposition in E with respect to $\mathfrak{M}$, if for every pair of Banach spaces $\tilde{P}, \tilde{R}$ ($\tilde{P}$ is a subspace in $\tilde{R}$), where $\tilde{P}$ is isomorphic to P , $\tilde{P} \in \mathfrak{M}$, $\tilde{R} \in \mathfrak{M}$, and an isomorphism φ of $\tilde{P}$ onto P , there exists a subspace R in E , $R \supset P$, and an isomorphism Φ of $\tilde{R}$ onto R such that: 1) $\Phi = \varphi$ on $\tilde{P}$; 2) $\|\Phi\| \leq a\|\varphi\|$, $\|\Phi^{-1}\| \leq a\|\varphi^{-1}\|$. If P is a subspace of a -universal disposition in E with respect to $\mathfrak{M}$ for every $a > 1$ (respectively for $a = 1$), then we shall call P a subspace of almost universal (respectively universal) disposition in E with respect to $\mathfrak{M}$.

Definition 3. We shall call E a space of a -universal ($a \geq 1$) (respectively almost universal, universal) disposition with respect to $\mathfrak{M}$, if every subspace P in E is a subspace of a -universal (respectively almost universal, universal) disposition in E with respect to $\mathfrak{M}$.

The main purpose of our note is to establish the existence of a separable Banach space of almost universal disposition with respect to $\mathfrak{R}$ (Theorem 2). In what follows we shall say that B_1 can be embedded in B_2 (B_1, B_2 are Banach spaces)

if there exists an isometry of B_1 onto some subspace $\tilde{B}_1$ of B_2 , and in doing so we shall identify B_1 with $\tilde{B}_1$.

Theorem 1. *Let P_1, E_1 and P_2, E_2 be two pairs of Banach spaces; P_1 is a subspace in E_1 ; P_2 is a subspace in E_2 , and φ is an isomorphism of P_1 onto P_2 . Then E_2 can be embedded in a Banach space B ,*

$$\dim B \leq \dim E_2 + \dim E_1/P_1$$

with a subspace $\tilde{E} \supset P_2$ such that there exists an isomorphism Φ of E_1 onto $\tilde{E}$, coinciding with φ on P_1 and satisfying the condition

$$\|\Phi\| = \|\varphi\|, \quad \|\Phi^{-1}\| = \|\varphi^{-1}\|.$$

* For the definition of the spaces C, c, m , see, for example, in ⁽¹⁾. A is the space of analytic functions in the disk $|z| < 1$, continuous for $|z| \leq 1$, with norm

$$\|f\| = \max_{|z|=1} |f(z)|.$$

We outline the proof of Theorem 1. Suppose first that φ is an isometry of P_1 onto P_2 .^{*} Consider the normed space of pairs (x, y) , $x \in E_1$, $y \in E_2$, with pairs of the form $(x + z, y - \varphi(z))$, $z \in P_1$, identified, and with norm

$$\|(x, y)\| = \inf_{z \in P_1} (\|x + z\| + \|y - \varphi(z)\|).$$

The completion of this space may be taken as B . If φ is an arbitrary isomorphism of P_1 onto P_2 and $P_2 = E_2$, then the proof of the theorem has in fact been given in ⁽³⁾ (pp. 211–212). The general case reduces to these two.

For what follows the following is essential.

Lemma 1. *Each of the following sets is a metric compactum:*

- 1) $\mathfrak{N}(k, n)$, k, n natural numbers ($k \leq n$), the set of all embedded pairs of Banach spaces $\{P, R\}$, $P \subset R$, $\dim P = k$, $\dim R = n$, with distance

$$\rho(\{P_1, R_1\}, \{P_2, R_2\}) = \ln \inf_T \|T\| \cdot \|T^{-1}\|,$$

where T ranges over the isomorphisms of R_1 onto R_2 which map P_1 onto P_2 (in this case pairs $\{P, R\}$, $\{P' R'\}$ for which $\rho(\{P, R\}, \{P' R'\}) = 0$ are identified).

- 2) $\mathfrak{G}(B)$ (B a finite-dimensional Banach space) —the set of all subspaces P in B with distance defined as the Hausdorff-Markus distance $\theta(P_1, P_2)$ ⁽⁴⁾.
- 3) $f(B_1, B_2, C)$ (B_1, B_2 n -dimensional Banach spaces, $C \geq 1$) —the set of all isomorphisms φ of B_1 onto B_2 such that $\max\{\|\varphi\|, \|\varphi^{-1}\|\} \leq C$, with distance

$$\nu(\varphi_1, \varphi_2) = \max\{\|\varphi_1 - \varphi_2\|, \|\varphi_1^{-1} - \varphi_2^{-1}\|\}.$$

For subspaces P and Q in E we shall consider the quantity

$$(P, Q) = \inf_{x \in P, \|x\|=1} \rho(x, Q).$$

From a result of Auerbach and Taylor (see, for example, ^(1,5)) the following follows.

Lemma 2. *If P is an n -dimensional subspace in a Banach space E , then there exists a direct complement Q to P in E such that*

$$(\widehat{P, Q}) \geq 1/n.$$

With the aid of Lemma 2 one establishes

Lemma 3. *Let P, R and $\widetilde{P}, \widetilde{R}$ be two pairs of Banach spaces; P a subspace in R ; $\widetilde{P}$ a subspace in $\widetilde{R}$; $\dim P = \dim \widetilde{P} = n$; Φ an isomorphism of R onto $\widetilde{R}$; φ the restriction of Φ to P , mapping P onto $\widetilde{P}$, with $\max\{\|\varphi\|, \|\varphi^{-1}\|\} \leq C < \infty$. For the given $\varepsilon > 0$ there is a $\delta = \delta(n, C, \varepsilon)$ such that, if for an isomorphism φ' of P onto $\widetilde{P}$ the condition $\nu(\varphi, \varphi') < \delta$ is fulfilled, then there exists an isomorphism Φ' of R onto $\widetilde{R}$, coinciding with φ' on P and such that $\nu(\Phi, \Phi') < \varepsilon$.*

It is easy to prove the following.

Lemma 4. *Let E' be a subspace of a Banach space E ; Q a direct complement to a subspace P in E' ; $(\widehat{P, Q}) = \beta > 0$; R a subspace in E such that $\tilde{\theta}(P, R) \leq \gamma < \beta$. Then Q is a direct complement to R in $R + Q$ ($R + Q$ is the closure of the set of elements of the form $x + y$, $x \in R$, $y \in Q$), and*

$$(\widehat{R, Q}) \geq \beta - \gamma.$$

With the aid of Lemmas 2 and 4 one establishes

Lemma 5. *Let P ($\dim P = n$) be a subspace of a -universal ($a \geq 1$) disposition in E with respect to $\mathfrak{R}$. For the given $\varepsilon > 0$ there is a $\delta = \delta(n, \varepsilon) > 0$ such that, if for some subspace $\widetilde{P}$ in E the condition $\theta(P, \widetilde{P}) < \delta$ is fulfilled, then $\widetilde{P}$ is a subspace of $(a + \varepsilon)$ -universal disposition in E with respect to $\mathfrak{R}$.*

Theorem 2. *There exists a separable Banach space of almost universal disposition with respect to $\mathfrak{R}$.*

* The proof given in this case belongs to M. I. Kadets. It is simpler than the author's original proof.

We outline the proof of Theorem 2. Consider sequences $\{\varepsilon_i\}_1^\infty$, $\{a_i\}_1^\infty$, $\varepsilon_i > 0$, $a_i > 1$, $\varepsilon_i \rightarrow 0$, $a_i \rightarrow \infty$, and the sets

$$\mathfrak{N}_i = \bigcup_{n=2}^{i+1} \bigcup_{k=1}^{n-1} \mathfrak{N}(k, n), \quad i = 1, 2, \dots$$

Let B_1 be a finite-dimensional Banach space; let $\{E_i^{(1)}\}_{i=1}^{m_1}$ be an ε_1 -net in $\mathfrak{S}(b_1)$; let $\{P_i^{(1)}, R_i^{(1)}\}_{i=1}^{n_1}$ be an ε_1 -net in $\mathfrak{N}_1$ ($P_i^{(1)}$ is a subspace of $R_i^{(1)}$); let $\{\varphi_k^{(i,j)}\}_{k=1}^{r_1}$ be an ε -net in $f(P_i^{(1)}, E_j^{(1)}, a_1)$, $i = 1, 2, \dots, n_1$, $j = 1, 2, \dots, m_1$ (which may also be empty). Applying Theorem 1 successively no more than $n_1 m_1 r_1$ times, we obtain the existence of a finite-dimensional Banach space $B_2 \supset B_1$ with the following property: for the isomorphism $\varphi_k^{(i,j)}$ there exists a subspace $F_k^{(i,j)}$ in B_2 , $F_k^{(i,j)} \supset E_j$, and an isomorphism $\Phi_k^{(i,j)}$ of $R_i^{(1)}$ onto $F_k^{(i,j)}$ such that $\Phi_k^{(i,j)}$ coincides with $\varphi_k^{(i,j)}$ on $P_i^{(1)}$ and

$$\|\Phi_k^{(i,j)}\| = \|\varphi_k^{(i,j)}\|, \quad \|\Phi_k^{-1(i,j)}\| = \|\varphi_k^{-1(i,j)}\|,$$

$$i = 1, 2, \dots, n_1; \quad j = 1, 2, \dots, m_1; \quad k = 1, 2, \dots, r_1.$$

If, starting from $\varepsilon_2, a_2, \mathfrak{N}_2, B_2$, we carry out the same reasoning, then the space $B_3 \supset B_2$ obtained thereby will again be finite-dimensional. Continuing this process without bound, we obtain a chain of embedded finite-dimensional spaces $B_1 \subset B_2 \subset B_3 \subset \dots$, and we may assume that, for some separable Banach space B , the B_i are subspaces of B , $i = 1, 2, \dots$, and $\bigcup_{i=1}^\infty B_i$ is dense in B . With the aid of Lemmas 3 and 5 it is verified that B is the required space.

Theorem 3. A space of almost universal disposition with respect to $\mathfrak{R}$ is almost universal with respect to $\mathfrak{H}$. A space of universal disposition with respect to $\mathfrak{R}$ is universal with respect to $\mathfrak{H}$.

Theorem 4. Let t_0 be the positive root of the equation $t(1+t^2)^{1/2} = 2$. For any topological space T and any $a < \frac{4}{\sqrt{2}}$, $C(T)^*$ is not a space of a -universal disposition with respect to $\mathfrak{R}$.

Define the class $\mathfrak{C}$ of infinite-dimensional Banach spaces by assuming that $B \in \mathfrak{C}$ if, for every finite-dimensional subspace P in B and $\varepsilon > 0$, there exists a subspace $E \subset B$, $E \supset P$, $\dim E = N = N(\varepsilon, P)$, and an isomorphism φ of E onto c^{N**} such that $\max\{\|\varphi\|, \|\varphi^{-1}\|\} < 1 + \varepsilon$. Thus, c, c_0, C, m are spaces of the class $\mathfrak{C}^{***}$.

Theorem 5. In a separable space of the class $\mathfrak{C}$ there exists a basis.

Corollary. In a separable space of almost universal disposition with respect to $\mathfrak{R}$ there exists a basis.

Theorem 6. For separable Banach spaces of almost universal disposition with respect to $\mathfrak{R}$ are almost isometric.

Definition 4. We shall call a Banach space E a -isotropic ($a \geq 1$) if, for any finite-dimensional subspaces P, Q , $\dim P = \dim Q$, and any isomorphism φ of P onto Q , there exists an isomorphism Φ of E onto itself such that $\Phi = \varphi$ on P , $\|\Phi\| \leq a\|\varphi\|$, $\|\Phi^{-1}\| \leq a\|\varphi^{-1}\|$. If E is a -isotropic for every $a > 1$ (respectively for $a = 1$), then we shall call E almost isotropic (respectively isotropic).

* Here $C(T)$ is the space of all real bounded continuous functions on T with norm $\|f\| = \sup_{x \in T} |f(x)|$.

** By c^n is denoted the space $C(T)$, where T is a collection of n points.

*** It can be shown that for any metric compactum T , $C(T) \in \mathfrak{C}$.

Theorem 7. A separable Banach space of almost universal disposition with respect to $\mathfrak{R}$ is almost isotropic.

Theorem 7 has some bearing on Mazur's problem on rotations of Banach spaces (¹, p. 211)).

Theorem 8. There does not exist a separable Banach space of universal disposition with respect to $\mathfrak{R}$.

I take this opportunity to express my sincere gratitude to M. I. Kadets for useful discussions.

Kharkov Automobile and Highway Institute

Received
18 I 1965

REFERENCES

- ¹ S. Banach, *Course of Functional Analysis*, Kiev, 1948.
- ² A. Pełczyński, *Studia Math.*, **24**, 285 (1964).
- ³ A. Pełczyński, *Studia Math.*, **19**, 209 (1960).
- ⁴ I. Ts. Gokhberg, A. S. Markus, *UMN*, **14**, no. 5, 135 (1959).
- ⁵ A. E. Taylor, *Bull. Am. Math. Soc.*, **53**, 614 (1947).
- ⁶ M. M. Day, *Normed Linear Spaces*, IL, 1959.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.