

# ALMOST-PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS IN HILBERT SPACE

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**Abstract**

**Full Text**

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*MATHEMATICS*

**V. V. ZHIKOV**

# ALMOST-PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS IN HILBERT SPACE

*(Presented by Academician I. G. Petrovsky on 27 IV 1965)*

I. Let

$$y(t) \sim \sum_{n=0}^{\infty} a_n e^{i\Lambda_n t}$$

be an almost-periodic function in the sense of Bohr. Denote by  $B_y$  the Banach space of almost-periodic functions obtained by taking the closure, in the uniform norm, of the linear span of the set  $\{e^{i\Lambda_n t}\}$ . Consider the operator  $L(y) = y'$ , defined on the set of almost-periodic functions from  $B_y$  having an almost-periodic derivative. This operator has a complete system of eigenfunctions of the form  $e^{i\Lambda_n t}$  with eigenvalues  $i\Lambda_n$ . From the fact that  $L(y)$  is the infinitesimal operator of the group of translations, one may conclude that its spectrum is exhausted by the closure of the set of eigenvalues, i.e. is purely point spectrum <sup>(1)</sup>. Hence, if  $|\Lambda_n| \geq \sigma$ , then there exists an inverse operator, i.e.

$$\int_0^t y(x) dx$$

is an almost-periodic function. This interesting fact was first discovered by Favard (<sup>(2)</sup>, p. 89) by means of multiplier theory for the Fourier transform. We shall consider an analogous problem for differential equations in Hilbert space.

Let  $\{A_k\}_{k=0}^n$  be self-adjoint, commuting operators in  $H$ ;  $S$  their joint spectrum, and  $f(t)$  an almost-periodic function with values in  $H$  and with Fourier-Bohr exponents  $\{\Lambda_m\}$ . For convenience we assume that  $A_n = E$ . Let  $\{c_k\}_{k=0}^n$  be complex numbers and  $c_n = 1$ .

1. **Strong problem on multipliers:** in the case of bounded operators  $\{A_k\}_{k=0}^{n-1}$ , find conditions under which there exists an almost-periodic function  $y(t)$ , having almost-periodic derivatives up to order  $n$  inclusive and satisfying the equation

$$\sum_{k=0}^n c_k A_k y^{(k)} = f(t). \quad (1)$$

2. **Weak problem on multipliers:** in the case of unbounded operators  $\{A_k\}_{k=0}^{n-1}$  with purely point spectrum, find conditions under which there exists an almost-periodic function  $y(t)$  satisfying equation (1) on a common complete system of eigenvectors of the operators  $\{A_k\}_{k=0}^{n-1}$ .

**Theorem 1.** *If  $|P(i\Lambda_n, s)| \geq \delta$ , where  $P(y, X) = P(y, x_0, \dots, x_{n-1}) = y^n + \sum_{k=0}^{n-1} c_k y^k x_k$  and  $s \in S$ , then the strong problem on multipliers is solvable.*

In the proof one uses the operational calculus for several closed operators with commuting resolvents. Ob-

denote by  $B_f(H)$  the Banach space of  $H$ -almost-periodic functions analogous to the space  $B_y$ , and let  $\{A_k\}_{k=0}^{n-1}$  have a purely point spectrum. Then, for the operator

$$C = \sum_{k=0}^n c_k A_k y^{(n)}$$

a complete system of eigenvectors is formally written down. The problem consists in proving that the spectrum of the operator  $C$  is purely point. We note that the weak problem on multipliers in the general form remains unsolved, since it has not been possible to study satisfactorily the spectral properties of a polynomial in several unbounded operators (the strong problem corresponds to a polynomial containing only one unbounded variable). It can be solved in the special case of the hyperbolic equation

$$u'' + Au = f, \quad (2)$$

where  $A$  is a positive definite operator with spectrum  $\{\lambda_n\}$ . The following result is obtained: if for all  $m$  and  $n$ ,  $|\sqrt{\lambda_n} - \Lambda_m| \geq \delta$ , then there exists a solution of equation (2) that is almost-periodic in the metric of the energy integral <sup>(3)</sup>.

When we pass to equations containing variable operators, the situation becomes much more complicated. However, there is a case in which the question of the existence of almost-periodic solutions is completely resolved—this is the case of a parabolic equation.

II. Let a linear parabolic equation be given,

$$u' + A(t)u + iB(t)u = f(t), \quad (3)$$

where  $A(t)$  are positive definite operators with a common domain of definition and completely continuous inverses,  $B(t)$  are symmetric, and  $f(t)$  is almost-periodic in  $H$ .

The expression  $\langle x, y \rangle$  will denote the scalar product of elements  $x$  and  $y$  in  $H$ . Suppose that all conditions for the existence and uniqueness of a generalized solution of equation (3), continuous in  $H$ , are satisfied <sup>(4)</sup>. In addition,

$$\langle A(t)x, x \rangle \geq \gamma \langle x, x \rangle, \quad c_1 \|A^{1/2}(0)x\| \leq \|A^{1/2}(t)x\| \leq c_2 \|A^{1/2}(0)x\|,$$

where the constants  $\gamma, c_1, c_2$  do not depend on  $t$ . Denote by  $W_0$  the Hilbert space with norm  $\|x\|_1^2 = \langle A^{1/2}(0)x, A^{1/2}(0)x \rangle$ , and let  $B_2$  be the Bochner space of strongly measurable functions on the interval  $(0, 1)$  with values in  $H$ . Let  $V$  be a linear space of continuous operators acting from  $W_0$  into  $H$ , endowed with the uniform operator topology. We assume that  $A(t)$  and  $B(t)$  are almost-periodic functions in  $V$ .

**Lemma 1.** Let  $x_n(t)$  be a sequence of functions on the interval  $(0, 1)$  such that

$$\int_0^1 \|Cx_n(\theta)\|^2 d\theta < K,$$

where  $C$  is a self-adjoint operator with a completely continuous inverse and  $K$  does not depend on  $n$ . If  $\langle x_n(t), y \rangle \rightarrow \langle x_0(t), y \rangle$  uniformly in  $t \in (0, 1)$  for every  $y \in H$ , then

$$x_n \xrightarrow{B_2} x_0.$$

**Theorem 2.** There exists a unique  $H$ -almost-periodic solution of equation (3).

We make several remarks concerning the proof. First of all, the existence of a bounded solution  $u(t)$  is obtained by Galerkin's method. This solution is unique, weakly uniformly continuous on the whole axis, and  $A(0)u(t)$  is a  $B_2$ -bounded function, i.e., the expression  $A(0)u(t+\theta)$ , considered for arbitrary  $t$  as an element of  $B_2$ , is a bounded function of  $t$ . The same is true for all limiting equations.

It follows from this that  $u(t)$  is weakly almost-periodic, by arguments going back to Favard <sup>(5)</sup>. The use of Lemma 1 gives the  $B_2$ -almost-periodicity of the function  $u(t)$ . Let us prove that almost-periodicity in  $H$  holds. For this it is enough to prove the compactness of the trajectory  $u(t)$ . Suppose the contrary. Then there is a sequence of real numbers  $\{\alpha_n\}$  and subsequences  $\{\alpha'_n\}$  and  $\{\alpha''_n\}$  such that

$$\begin{aligned}
 u_n(\theta + \alpha_n - 1) &\xrightarrow{B_2} \bar{u}(\theta), \\
 A(t + \alpha_n - 1) &\xrightarrow{V} \hat{A}(t) \quad \text{uniformly in } t, \\
 B(t + \alpha_n - 1) &\xrightarrow{V} \hat{B}(t) \quad \text{uniformly in } t, \\
 f(t + \alpha_n - 1) &\xrightarrow{H} \hat{f}(t) \quad \text{uniformly in } t;
 \end{aligned} \tag{4}$$

$$\|u(\alpha'_n) - u(\alpha''_n)\| \geq \delta \quad (\delta \text{ does not depend on } n). \tag{5}$$

Denote

$$\begin{aligned}
 \omega_n(t) &= u(t + \alpha'_n - 1) - u(t + \alpha''_n - 1), \\
 g_n &= f(t + \alpha'_n - 1) - f(t + \alpha''_n - 1), \\
 G_n &= A(t + \alpha'_n - 1) - A(t + \alpha''_n - 1)
 \end{aligned}$$

and

$$F_n = B(t + \alpha'_n - 1) - B(t + \alpha''_n - 1).$$

After simple calculations, for  $t_1 \in [0, 1]$  we have

$$\begin{aligned}
 &\frac{1}{2}\|\omega_n(1)\|^2 - \frac{1}{2}\|\omega_n(t_1)\|^2 + \int_{t_1}^1 \|A^{1/2}(\alpha'_n + \theta - 1)\omega_n(\theta)\|^2 d\theta \\
 &+ \operatorname{Re} \int_{t_1}^1 \langle G_n(\theta)u(\alpha''_n + \theta - 1), \omega_n(\theta) \rangle d\theta \\
 &+ \operatorname{Im} \int_{t_1}^1 \langle F_n(\theta)u(\alpha''_n + \theta - 1), \omega_n(\theta) \rangle d\theta \\
 &= \operatorname{Re} \int_{t_1}^1 \langle g_n(\theta), \omega_n(\theta) \rangle d\theta.
 \end{aligned}$$

The third term of this equality is positive; the fourth and fifth terms and the right-hand side can be made arbitrarily small by choosing  $n$ . Hence, using (5), for every  $t_1 \in [0, 1]$  and sufficiently large  $n$  we have

$$\|\omega_n(t_1)\|^2 \geq \|\omega_n(1)\|^2 - \frac{1}{2}\delta^2 \geq \frac{1}{2}\delta^2.$$

But this contradicts the fact that, by (4),  $\omega_n \xrightarrow{B_2} 0$ . The theorem is proved.

**Remark.** The assertion of Theorem 2 remains valid if, instead of the  $H$ -almost-periodicity of  $f(t)$ , one assumes the  $H$ -almost-periodicity of the function  $A^{-1/2}(0)f(t)$ . This is clearly seen in the example of the simplest equation

$$u' + Au = f,$$

where  $A$  is a positive definite operator independent of  $t$ . The desired solution is given by the formula

$$u(t) = \int_{-\infty}^t e^{-A(t-s)} f(s) ds = \int_{-\infty}^t A^\theta e^{-A(t-s)} A^{-\theta} f(s) ds.$$

Since

$$\int_{-\infty}^t \|A^{\theta_0} e^{-A(t-s)}\| ds < \infty, \quad \text{if } \theta_0 < 1,$$

it is sufficient that  $A^{-\theta_0} f(t)$  be an  $H$ -almost-periodic function.

The arguments used in the proof of Theorem 2 also make it possible to study the Navier–Stokes equation in a bounded domain  $\Omega$

$$\partial V / \partial t = \nu \Delta V - v_k V_{x_k} + \text{grad } p + F(t), \quad V_s = 0, \quad \text{div } V = 0, \quad (6)$$

with an  $L_2$ -almost-periodic function  $F(t)$ .

**Theorem 3.** There exists a constant  $c_1(V, \Omega)$  such that, if

$$\sup_{-\infty < t < \infty} \|F(t)\|_{L_2} \leq c_1,$$

then there exists a unique  $L_2$ -almost-periodic solution of equation (6).

Here the following simple observation is used.

**Lemma 2.** There exists a constant  $c_2(V, \Omega)$  such that, if

$$\|V(0)\|_{W_2^1} \leq c_2, \quad \sup_{-\infty < t < \infty} \|F(t)\|_{L_2} \leq c_2,$$

then Hopf's weak solution is unique and is a continuous function in  $W_2^1$ .

We note that the two-dimensional case of equation (7) has been studied in detail in <sup>(6,7)</sup>, with essential use of the uniqueness of Hopf's weak solution.

The parabolic equation turns out to be especially convenient for singling out periodic solutions. We shall dwell on one simple theorem in this circle of questions. Let the equation

$$x' + Ax = Q(x, t), \quad (7)$$

be given, where  $A$  is a positive definite operator with completely continuous inverse and  $Q(x, t)$  is a nonlinear operator, periodic in  $t$  with period  $\omega$ . Introduce

the Hilbert spaces  $H_\theta$  with norm  $\|x\|_\theta^2 = \langle A^\theta x, A^\theta x \rangle$  for  $\theta < 1$ . Suppose that  $Q(x, t)$  is a continuous operator acting from  $H_{\theta_0}$  into  $H$ . Obviously, the operator  $R(x, t) = A^{-\theta_0} Q(x, t)$  is a continuous operator in  $H_{\theta_0}$ . Let  $T_r$  be the ball of radius  $r$  in  $H_{\theta_0}$  with center at zero.

**Theorem 4.** If there exist positive numbers  $r_1, r_2$  such that, for every  $t$ ,  $R(x, t) \in T_{r_2}$  whenever  $x \in T_{r_1}$ , and

$$r_2/r_1 \leq c_0(A, \theta_0) = \int_{-\infty}^0 \|A^{\theta_0} e^{-A\alpha}\| d\alpha,$$

then there exists at least one  $\omega$ -periodic solution of equation (7).

Theorem 4 is proved by applying Schauder's principle. Its finite-dimensional analogue is well known <sup>(8)</sup>.

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## REFERENCES

- <sup>1</sup> Yu. I. Lyubich, V. I. Matzaev, DAN, **131**, No. 1 (1960).
- <sup>2</sup> B. M. Levitan, *Almost-periodic functions*, Moscow, 1953.
- <sup>3</sup> V. V. Zhikov, DAN, **163**, No. 3 (1965).
- <sup>4</sup> O. A. Ladyzhenskaya, *Mathematical Collection*, **39** (81), 491 (1956).
- <sup>5</sup> J. Favard, *Leçons sur les fonctions presque périodiques*, Paris, 1933.
- <sup>6</sup> G. Prouse, Rend. Sem. Mat. Padova, **33** (1963).
- <sup>7</sup> C. Foias, Rend. Sem. Mat. Padova, **32** (1962).
- <sup>8</sup> M. A. Krasnosel'skii, A. I. Perov, DAN, **126**, No. 1 (1959).

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