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1965

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Abstract

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UDC 517.544.3

MATHEMATICS

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ON THE UNIQUE SOLVABILITY OF PLANE PROBLEMS OF GAS DYNAMICS WITH FREE BOUNDARIES

(Presented by Academician I. N. Vekua on 15 III 1965)

In the present paper we prove the existence and uniqueness of subsonic potential gas flows for a broad class of problems of gas dynamics with free boundaries, among which, for example, the following problems may be indicated:

1. Problems of flow with separation according to Kirchhoff's scheme past a given (unknown) curvilinear obstacle by an unbounded gas stream and by a gas stream in a channel with a given (unknown) shape of the walls. The shape of the unknown part of the boundary L_z of the flow domain is determined by the prescribed value on it of the speed $q = q(x)$ (on the jets $q = \text{const}$).
2. The inverse and mixed problem of determining the shape of an airfoil profile L_z from the chord diagram, when on the unknown part of the profile (or on all of it in the inverse problem) the value of the speed $q = q(x)$ is prescribed. In this problem the solution may also correspond to a nonclosed contour L_z .

The basis of the method we propose for investigating problems of gas dynamics is the apparatus of generalized analytic functions, well developed by I. N. Vekua ⁽¹⁾.

For an incompressible fluid in problem 1, which has already become classical, the principal results were obtained by M. A. Lavrent'ev ⁽²⁾, Leray ⁽³⁾, and a number of other authors. Problems 1 and 2 in the case of an incompressible fluid were investigated in the author's papers ^(4, 5).

Let $\theta = \arctg \varphi_y / \varphi_x$ be the angle of inclination of the flow velocity ($\theta_\infty = 0$), and suppose that on the prescribed parts of the boundary L_z , $\theta(x) \in C_\alpha^1$ and

$$|\theta(x)| < \pi/2 - \varepsilon_0 \quad (\varepsilon_0 > 0). \quad (1)$$

On the sought parts of the boundary L_z , where the value of the speed $q(x) \in C_\alpha^1$ is prescribed, it is assumed that the inequality $|\ln q(x)| < \infty$ is satisfied, which is equivalent to

$$q^*(x) = \int_1^{q(x)} \frac{\sqrt{1-M^2}}{q} dq < N < \infty, \quad (2)$$

where $1-M^2 = \frac{1}{\rho} \frac{d(q\rho)}{dq}$ and $\rho = \rho(q)$ is the density.

In view of conditions (1) and (2), at the branch points of the flow the boundary L_z has a “cusp” directed toward the flow, and the channel walls are parallel at infinity.

As is known ⁽⁶⁾, the equations of gas dynamics in the case of a plane potential flow can be written in the form

$$\rho(q)\varphi_x = \psi_y; \quad \rho(q)\varphi_y = -\psi_x,$$

where $q = (\varphi_x^2 + \varphi_y^2)^{1/2}$ is the magnitude of the flow velocity.

The flow is assumed throughout to be subsonic ⁽⁶⁾, i.e., the ellipticity of the system of equations of gas dynamics is assumed for all q ($1-M^2 > 0$). From this assumption, in particular, it follows that fulfillment of inequalities (1), (2) on the entire boundary L_z entails their equ-

numbered fulfillment throughout the domain D_z . In the plane of the variables x, ψ , for the functions $q^*(x, \psi)$ and $\theta(x, \psi)$ one obtains the complex equation

$$\omega_{\bar{\tau}} - \mu_1(\omega)\omega_\tau = 0, \quad (3)$$

where $\omega = q^* - i\theta$, $\tau = x + i\psi$,

$$\mu_1(\omega) = \frac{\rho q - \cos \theta \sqrt{1-M^2} + i \sin \theta}{\rho q + \cos \theta \sqrt{1-M^2} - i \sin \theta}.$$

The system of equations corresponding to the complex equation (3) is elliptic provided inequalities (1) and (2) are fulfilled on the entire boundary L_z . In the case of incompressibility of the fluid, $\rho = 1$, $M = 0$, $\omega = \ln q - i\theta$ and $\mu_1(\omega) = (e^\omega - 1)/(e^\omega + 1)$ ^(4,5).

In problems 1 and 2, in the plane $\tau = x + i\psi$, the image D_τ of the flow domain D_z is, obviously, known; moreover, the fulfillment of inequalities (1), (2) on the entire boundary ensures the quasiconformality of the mapping $z = z(\tau)$. In jet problems this is the plane or a strip with a cut along a semi-infinite interval of

the real axis; in problems of determining an airfoil profile it is the plane with a cut along a finite segment of the real axis.

Map conformally D_τ onto the interior of the circle $|\zeta| < 1$ in such a way that the image of the unknown part of the boundary L_z is mapped into the upper semicircle, and that of the prescribed part into the lower one. Then L_z is completely unknown, and the normalization of the mapping is arbitrary. Then equation (3) is transformed into the form

$$\omega_{\bar{\zeta}} - \mu(\omega, \zeta)\omega_{\zeta} = 0, \quad (4)$$

where $\mu(\omega, \zeta) = \mu_1(\omega) \exp\{-2i \arg(d\tau/d\zeta)\}$. For a completely unknown boundary L_z , for equation (4) we obtain the Dirichlet boundary-value problem

$$\operatorname{Re} \omega(e^{i\gamma}) = q^*[x(\gamma)], \quad \gamma \in [0, 2\pi]; \quad \operatorname{Im} \omega(1) = 0; \quad (5)$$

but if a part of L_z is prescribed, we obtain the mixed boundary-value problem

$$\operatorname{Re} \omega(e^{i\gamma}) = q^*[x(\gamma)], \quad \gamma \in [0, \pi]; \quad \operatorname{Im} \omega(e^{i\gamma}) = -\theta[x(\gamma)],$$

$$\gamma \in [\pi, 2\pi]. \quad (6)$$

From the conditions on the functions $q^*(x)$ and $\theta(x)$ it follows that $\omega(z) \in C_\alpha^1$ on L_z , except, possibly, for the junction points of the unknown and prescribed parts of L_z , in neighborhoods of which $\omega(z) \in C_\alpha$. The function $\omega(\zeta)$ has an analogous property. From this, in particular, follows the fulfillment of inequality (1) on the entire boundary L_z .

We shall prove the uniqueness of the solutions of problems 1 and 2 satisfying inequalities (1) and (2) on the entire boundary L_z . For this it is obviously sufficient to show the uniqueness of the function $\omega(\zeta)$ solving problem (4), (5) or (4), (6) under the fulfillment of the inequality $|\mu(\omega, \zeta)| \leq \mu_0 < 1$. Suppose there are two such functions ω^1 and ω^2 . Then $\omega = \omega^1 - \omega^2$ satisfies homogeneous boundary conditions and the equation

$$\omega_{\bar{\zeta}} - \mu(\omega^1, \zeta)\omega_{\zeta} + B(\zeta)\omega = 0, \quad (7)$$

where

$$B(\zeta) = \omega_{\zeta}^2(\omega^2 - \omega^1)^{-1}[\mu(\omega^2, \zeta) - \mu(\omega^1, \zeta)]$$

for $\omega^1 - \omega^2 \neq 0$, and $B(\zeta) = 0$ for $\omega^1 = \omega^2$. Since $\omega \in L_p$ ($p > 2$), and the function $\mu(\omega, \zeta)$ is differentiable with respect to the argument ω , it follows that

$B(\zeta) \in L_p$ ($p > 2$). Let $\chi = \chi(\zeta)$ be a homeomorphism of the circle K ($|\zeta| < 1$) onto the circle $|\chi| < 1$, satisfying the Beltrami equation $\chi_{\bar{\zeta}} - \mu(\omega^1, \zeta)\chi_{\zeta} = 0$ and leaving the points ± 1 in place. Denote

$$B(\zeta) [\bar{\chi}_{\bar{\zeta}} - \mu\chi_{\zeta}]^{-1} = c(\chi) \in L_p \quad (p > 2)$$

and

$$-\frac{1}{\pi} \iint_K \left[\frac{c(t)}{t - \chi} + \frac{\chi c(\bar{t})}{1 - \chi\bar{t}} \right] dK = \tilde{T}_1 c.$$

Then any solution of equation (7) can be represented in the form

$$\omega = \Phi(\chi) \exp\{\tilde{T}_1 c\} \quad (1),$$

where $\Phi(\chi)$ is an analytic function.

Since $\text{Im } \tilde{T}_1 c = 0$ on the circle $|\chi| = 1$, the analytic function $\Phi(\chi)$ satisfies the homogeneous problem (5) or (6) and is bounded in the closed disk $|\chi| \leq 1$; consequently, $\Phi(\chi) \equiv 0$, whence $\omega = \omega^1 - \omega^2 \equiv 0$.

Let us now prove the existence of solutions in problems 1 and 2. We shall assume that, on the unknown part L_z , one of the conditions $q = \text{const}$ or $q = q(x)$ is fulfilled, with $|dq/dx| < \infty$, $q(x) \in C_{\alpha,1}$, for which satisfaction of inequalities (1) and (2) on the whole boundary L_z is shown. We include problems (4), (5) and (4), (6) in a one-parameter family of problems by introducing in conditions (5) and (6), before $q^*[x(\gamma)]$ and $-\theta[x(\gamma)]$, the multiplier λ . The value $\lambda = 0$ corresponds to the trivial solution $\omega(\zeta) \equiv 0$, and $\lambda = 1$ to the solution of the original boundary-value problems.

Consider the operators

$$T_1 f = \frac{1 - \zeta}{\pi} \iint_K \left[\frac{f(t)}{(t - 1)(t - \zeta)} + \frac{\overline{f(t)}}{(1 - \bar{t})(1 - \zeta\bar{t})} \right] dK,$$

$$T_2 f = -\frac{R(\zeta)}{\pi} \iint_K \left[\frac{f(t)}{R(t)(t - \zeta)} - \frac{\overline{f(t)}}{\overline{R(t)}(1 - \zeta\bar{t})} \right] dK,$$

where $f(t) \in L_p$, and $R(\zeta) = \sqrt{i(\zeta^2 - 1)}$. The well-studied properties, due to I. N. Vekua, of the operators $\tilde{T}_1$ and T_1 were used by us as the basis for the construction and study of the operator T_2 ⁽⁵⁾. The operators T_1 , T_2 are completely continuous in L_p ($p > 2$); $\partial T_i f / \partial \bar{\zeta} = f$ ($i = 1, 2$) for any function $f \in L_p$; $T_i f$ satisfy, respectively, the homogeneous conditions (5) and (6), while the operators $S_i f = \partial T_i f / \partial \zeta$ are bounded in L_p and $\|S_i\|_{L_2} = 1$. We shall seek the solution of the family of boundary-value problems (4), (5) and (4), (6) in

the form: $\omega(\zeta, \lambda) = \lambda\Phi_i(\zeta) + T_i f$, where $f \in L_p$ ($p > 2$) and $\Phi_i(\zeta)$ ($i = 1, 2$) are analytic functions satisfying, respectively, the boundary conditions (5) and (6). Then, for the function $f(\zeta)$, the singular integral equations are obtained

$$f - \mu(\zeta, \lambda\Phi_i + T_i f)S_i f - \lambda\mu(\zeta, \lambda\Phi_i + T_i f) d\Phi_i/d\zeta = 0. \quad (8)$$

Denote by G the set of functions $g(\zeta)$ for which $\Omega_i = \lambda\Phi_i + T_i g$ satisfy, in the disk $|\zeta| \leq 1$, the conditions $|\operatorname{Re} \Omega_i| < N < \infty$ and $|\operatorname{Im} \Omega_i| < \pi/2 - \varepsilon_0$. Let $g(\zeta)$ and $f(\zeta)$ be arbitrary functions of the set G . Then

$$\|\mu(\eta, \Omega_i)S_i f\|_{L_p} \leq \mu_0[1 + \delta(p)]\|f\|_{L_p},$$

where $\delta(p) \rightarrow 0$ as $p \rightarrow 2$, and therefore $\mu_0(1 + \delta) < 1$ for p sufficiently close to 2.

Thus equation (8) can be written in the form

$$f - A_i(\Omega_i, \lambda) = 0, \quad (9)$$

where $A_i(\Omega_i, \lambda) = \lambda(E - \mu S_i)^{-1} \mu(d\Phi_i/d\zeta)$. From the continuity of the operators A_i in $\Omega_i = \lambda\Phi_i + T_i g$ and the complete continuity of the operators T_i , it follows that $A_i(\lambda\Phi_i + T_i g, \lambda)$ is completely continuous in g for any fixed λ , and continuous in λ for arbitrary fixed $g \in G$. Thus we are in the conditions of applicability to the operators A_i of the Leray–Schauder method ⁽⁷⁾.

It is evident that the method proposed by us also yields existence and uniqueness theorems for problems of flow in channels with completely prescribed wall shape and of flow around a prescribed profile.

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Received
9 III 1965

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