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1965

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Fig. 1

Figure 1: Fig. 1

Abstract

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UDC 539.17.01

PHYSICS

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INTEGRAL REPRESENTATION FOR THE TWO-PARTICLE AMPLITUDE IN THE NON-RELATIVISTIC CASE

(Presented by Academician N. N. Bogolyubov, January 4, 1965)

1. In the work of L. D. Faddeev ⁽¹⁾ a system of coupled equations was proposed that is suitable for describing the three-body problem when the particles interact with one another by pairwise forces. A remarkable feature of these equations is, first, that in solving them the three-body scattering amplitude T is expressed in terms of the two-particle amplitudes T_{ij} , and, second, that one obtains comparatively easily an expansion for T in a perturbation-theory series and, as was shown in ⁽²⁾, one can formulate the corresponding diagram technique. Diagrams of this kind are completely determined by specifying vertices, which are two-particle amplitudes off the energy surface; therefore, in order to study the analytic properties of these diagrams it is essential to know the analytic properties of T_{ij} .

Fig. 1

2. The purpose of the present work is to give an integral representation for the two-particle amplitude off the energy surface in the nonrelativistic case. Let us write the matrix element of the operator t in the form:

$$\delta(\mathbf{p}_i + \mathbf{k}_i - \mathbf{p}_f - \mathbf{k}_f) \left\langle (\mathbf{p}_f - \mathbf{k}_f)/2 \left| t \left(z - \frac{(\mathbf{p}_f + \mathbf{k}_f)^2}{2} \right) \right| (\mathbf{p}_i - \mathbf{k}_i)/2 \right\rangle. \quad (1)$$

The operator t satisfies the Lippmann–Schwinger equation

$$t = V - VG_0 t, \quad (2)$$

where V is the interaction potential depending on the difference of coordinates (for convenience in what follows a Yukawa potential is chosen), and G_0 is the Green function, having the form (in a system of units in which $\hbar^2 = 2m_1 = 2m_2 = 1$)

$$G_0 = \frac{1}{\mathbf{k}^2 - [z - (\mathbf{p}_f + \mathbf{k}_f)^2/2]}.$$

The independence of the combinations $(\mathbf{p} + \mathbf{k})/2$ and $(\mathbf{p} - \mathbf{k})/2$ makes it possible to fix the expression $E = z - (\mathbf{p}_f + \mathbf{k}_f)^2/2$, and then the matrix element (1) will depend on three scalar squares: $((\mathbf{p}_f - \mathbf{k}_f)/2)^2$, $((\mathbf{p}_i - \mathbf{k}_i)/2)^2$, $(\mathbf{k}_f - \mathbf{k}_i)^2$. Iterating equation (2), we obtain a perturbation-theory series whose corresponding diagrams are depicted in Fig. 1.* The contribution from each diagram in the α -representation (up to constant factors) can be written in the form of the integral ⁽⁴⁾:

$$F_n(\mathbf{q}_i) \sim \int_0^\infty \frac{\prod_{i=1}^{n-1} d\alpha_i \prod_{j=0}^{n-1} d\beta_{j,j+1}}{D^{3/2}(\alpha, \beta)} \exp \left[\frac{R(\alpha, \beta, \mathbf{q}_1, \mathbf{q}_2)}{D(\alpha, \beta)} - \left(\sum_{i=1}^{n-1} \alpha_i \right) E + \left(\sum_{j=0}^{n-1} \beta_{j,j+1} \right) \mu^2 \right], \quad (3)$$

where, for convenience, the notation $\mathbf{q}_1 = (\mathbf{p}_f - \mathbf{k}_f)/2$ and $\mathbf{q}_2 = (\mathbf{p}_i - \mathbf{k}_i)/2$ has been introduced.

* Nonrelativistic diagrams are usually drawn in a different form, namely: the thin lines do not emerge from a single point a , as in Fig. 1, but from separate points. We shall agree that at the vertex a we shall write the interaction constant to the power whose value is equal to the number of incoming lines. Thus, the first diagram in Fig. 1 will be of second order, the second of fourth order, etc.

In (3), $D(\alpha, \beta)$ is the determinant of the matrix

$$d = \begin{pmatrix} \beta_{01} + \alpha_1 + \beta_{12} & -\beta_{21} & 0 & \dots \\ -\beta_{12} & \beta_{12} + \alpha_2 + \beta_{23} & -\beta_{23} & \dots \\ 0 & -\beta_{23} & \beta_{23} + \alpha_3 + \beta_{34} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}. \quad (4)$$

$R(\alpha, \beta, \mathbf{q}_1, \mathbf{q}_2)$ is the bordered determinant of the determinant $D(\alpha, \beta)$:

$$R(\alpha, \beta, \mathbf{q}_1, \mathbf{q}_2) = \begin{vmatrix} 0 & B_i \\ B_i & d \end{vmatrix}, \quad \text{where } B_1 = -\beta_{01}\mathbf{q}_1, \quad B_{n-1} = -\beta_{n-1,n}\mathbf{q}_2, \quad B_i = 0,$$

$i = 2, 3, \dots, n-2$. It is easy to verify that (4)

$$\min_{k_\nu} \left[\sum_{i=1}^{n-1} \alpha_i (\mathbf{k}_i^2 - E) + \sum_{j=0}^{n-1} \beta_{j,j+1} (\mathbf{k}_j^2 + \mu^2) \right] = \frac{R(\alpha, \beta, \mathbf{q}_1, \mathbf{q}_2)}{D(\alpha, \beta)} - \left(\sum_{i=1}^{n-1} \alpha_i \right) E + \left(\sum_{j=0}^{n-1} \beta_{j,j+1} \right) \mu^2 = A_1(\alpha, \beta) \mathbf{q}_1^2 -$$

Here the vectors $\mathbf{q}_{1,2}$; $\mathbf{k}_i$, $\mathbf{k}_j$ satisfy the conservation law at each vertex. The minimization is performed over the independent momenta that remain after integration over the δ -functions;

$$\begin{aligned} A_1(\alpha, \beta) &= \beta_{01} + \frac{1}{D(\alpha, \beta)} \begin{vmatrix} 0 & \beta_{01} & 0 \dots 0 \\ \beta_{01} & & \\ \vdots & & d \\ \beta_{n-1,n} & & \end{vmatrix}, \\ A_2(\alpha, \beta) &= \beta_{n-1,n} + \frac{1}{D(\alpha, \beta)} \begin{vmatrix} 0 & 0 \dots \beta_{n-1,n} \\ \beta_{01} & \\ \vdots & d \\ \beta_{n-1,n} & \end{vmatrix}, \quad (6) \\ A_3(\alpha, \beta) &= -\frac{1}{D(\alpha, \beta)} \begin{vmatrix} 0 & 0 \dots \beta_{n-1,n} \\ \beta_{01} & \\ \vdots & d \\ 0 & \end{vmatrix}. \end{aligned}$$

We note that (3) $D(\alpha, \beta) \geq 0$; $A_i(\alpha, \beta) \geq 0$, $i = 1, 2, 3$.

3. Theorem. *The analyticity domain of any diagram of order $2n > 4$ (n is the number of thin lines in the diagram) includes in itself (or coincides with) the analyticity domain of the fourth-order diagram, provided only that $E = z - (\mathbf{p}_f + \mathbf{k}_f)^2/2 < 0$ ($-E = m^2$).*

Proof. Let, for the functions Q_{2n} and $Q_{2n'}$ corresponding to diagrams of orders $2n$ and $2n'$ (where $n < n'$), the inequality

$$Q_{2n'} \geq Q_{2n}; \quad (7)$$

hold; then we shall say that the diagram of order $2n$ majorizes the diagram of order $2n'$, and these words have the following meaning: the analyticity domain $G_{n'}$ of the diagram of order $2n'$ contains in itself (or coincides with) the analyticity domain G_n of the diagram of order $2n$, i.e. $G_n \subseteq G_{n'}$.

Let the diagram of order $2n'$ contain the figure shown in Fig. 2a, and let the diagram of order $2n$ be obtained from it by replacing figure 2a by figure 2b. According to (5), in order to establish inequality (1) under such a replacement it is sufficient to show that the minimum with respect to $\mathbf{k}$ of $f(\alpha, \beta, \mathbf{k}, \mathbf{q}_i)$ is greater than $\varphi(\gamma, \mathbf{q}_i)$, with an appropriate choice of γ , where

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

$$f(\alpha, \beta, \mathbf{k}, \mathbf{q}_i) = \beta_1(\mathbf{q}_1^2 + m^2) + \beta_2(\mathbf{q}_2^2 + m^2) + \alpha_1(\mathbf{k}^2 + m^2) + \alpha_2[(\mathbf{q}_1 - \mathbf{k})^2 + \mu^2] + \alpha_3[(\mathbf{q}_2 - \mathbf{k})^2 + \mu^2],$$

$$\varphi(\gamma, \mathbf{q}_i) = \gamma_1(\mathbf{q}_1^2 + m^2) + \gamma_2(\mathbf{q}_2^2 + m^2) + \gamma_3[(\mathbf{q}_1 - \mathbf{q}_2)^2 + \mu^2].$$

Indeed:

$$\min_{\mathbf{k}} f(\alpha, \beta, \mathbf{k}, \mathbf{q}_i) = \left(\beta_1 + \frac{\alpha_1 \alpha_2}{\sum_{i=1}^3 \alpha_i} \right) (\mathbf{q}_1^2 + m^2) + \left(\beta_2 + \frac{\alpha_1 \alpha_3}{\sum_{i=1}^3 \alpha_i} \right) (\mathbf{q}_2^2 + m^2) + \frac{\alpha_2 \alpha_3}{\sum_{i=1}^3 \alpha_i} [(\mathbf{q}_1 - \mathbf{q}_2)^2 + \mu^2] + \frac{\alpha_1^2}{\sum_{i=1}^3 \alpha_i} m^2$$

and if we define

$$\gamma_1 = \beta_1 + \alpha_1 \alpha_2 / \sum_{i=1}^3 \alpha_i; \quad \gamma_2 = \beta_2 + \alpha_1 \alpha_3 / \sum_{i=1}^3 \alpha_i; \quad \gamma_3 = \alpha_2 \alpha_3 / \sum_{i=1}^3 \alpha_i,$$

then

$$\min_{\mathbf{k}} f(\alpha, \beta, \mathbf{k}, \mathbf{q}_i) \geq \varphi(\gamma, \mathbf{q}_i). \quad (8)$$

When one of the two $\mathbf{q}_i$ (say, $\mathbf{q}_1$) corresponds to an external line, the inequalities (8) are satisfied if $\mathbf{q}_1^2 > -(m + \mu)^2$.

Applying the indicated substitution to the diagram of Fig. 1, we arrive at the proof of the theorem. Thus, in order to find the minimal real domain of holomorphy of the amplitude, it suffices to restrict ourselves to the study of the analytic properties of the fourth-order diagram.

Fig. 2

Fig. 3

4. The analytic properties of the triangular diagram in field theory are well known ⁽⁴⁻⁵⁾. Omitting the calculations, we present the results obtained. The domain in which $Q > 0$ is determined by the inequalities

$$\xi_i > 0, \quad i = 1, 2, 3; \quad \xi_i + \xi_j < 0; \quad \xi_k > \xi_i \xi_j - \sqrt{(1 - \xi_i^2)(1 - \xi_j^2)}, \quad (9)$$

where the ξ_i are given by the relations

$$\xi_1 = (m^2 + \mu^2 + z_1)/2m\mu, \quad \xi_2 = (m^2 + \mu^2 + z_2)/2m\mu, \quad \xi_3 = (2\mu^2 + z_3)/2\mu^2$$

(Fig. 3).

To obtain the integral representation, we shall carry out a study analogous to the works ^(3,6). Denote by G_0 the domain (9). Following By (7), one can find a certain complex domain of analyticity G over the prescribed real domain of analyticity G_0 . Indeed, let G denote the set of complex points $z_i = x_i + iy_i$ ($i = 1, 2, 3$), for which there exists a real λ such that $x_i + \lambda y_i \in G_0$. Then, from the linearity of Q with respect to z_i , it follows that $Q \neq 0$, which proves our assertion. The domain G is a natural domain of holomorphy in the space z_i ($i = 1, 2, 3$) ^(6,7).

In what follows it is convenient to rewrite (3) in a somewhat different (but identical) form:

$$F_n(z_i) \sim \int_0^1 \dots \int_0^1 \frac{\delta \left(1 - \sum_{i=1}^{n-1} \alpha'_i - \sum_{j=0}^{n-1} \beta'_{j,j+1} \right) \prod_{i=1}^{n-1} d\alpha'_i \prod_{j=0}^{n-1} d\beta'_{j,j+1}}{D^{3/2}(\alpha', \beta') [Q(\alpha', \beta', z_i, m^2, \mu^2) - i0]^{(n+1)/2}}. \quad (10)$$

where $(n+1)$ is the number of vertices in the diagram. It is expedient to consider separately the diagrams for even n and odd n .

We denote the corresponding functions by $F_{2k}(z_i)$ and $F_{2k+1}(z_i)$. Let

$$\eta_i = \frac{A_i(\alpha', \beta')}{\sum_{i=1}^3 A_i(\alpha', \beta')}, \quad i = 1, 2, 3; \quad \rho = \frac{\left(\sum_{i=1}^{n-1} \alpha'_i \right) m^2 + \left(\sum_{j=0}^{n-1} \beta'_{j,j+1} \right) \mu^2}{\sum_{i=1}^3 A_i(\alpha', \beta')}.$$

Substituting into (10) the identity

$$\begin{aligned} 1 &= \int_0^1 d\eta_1 \int_0^1 d\eta_2 \int_0^1 d\eta_3 \int_{-\infty}^{\infty} d\rho \delta \left(\eta_1 - \frac{A_1(\alpha', \beta')}{\sum_{i=1}^3 A_i(\alpha', \beta')} \right) \delta \left(\eta_2 - \frac{A_2(\alpha', \beta')}{\sum_{i=1}^3 A_i(\alpha', \beta')} \right) \\ &\quad \times \delta \left(\eta_3 - \frac{A_3(\alpha', \beta')}{\sum_{i=1}^3 A_i(\alpha', \beta')} \right) \delta \left(\rho - \frac{\left(\sum_{i=1}^{n-1} \alpha'_i \right) m^2 + \left(\sum_{j=0}^{n-1} \beta'_{j,j+1} \right) \mu^2}{\sum_{i=1}^3 A_i(\alpha', \beta')} \right) \end{aligned}$$

and integrating with respect to ρ k times by parts, we obtain the desired integral representations

$$F_{2k}(z_i) = \int_0^1 \int_0^1 \int_0^1 d\eta_1 d\eta_2 d\eta_3 \int_{\rho_0(\eta_i)}^{\infty} d\rho \frac{f_{2k}(\eta_i, \rho) \delta \left(1 - \sum_{i=1}^3 \eta_i \right)}{[\eta_1 z_1 + \eta_2 z_2 + \eta_3 z_3 + \rho - i0]^{1/2}}, \quad (11a)$$

$$F_{2k+1}(z_i) = \int_0^1 \int_0^1 \int_0^1 d\eta_1 d\eta_2 d\eta_3 \int_{\rho_0(\eta_i)}^\infty d\rho \frac{f_{2k+1}(\eta_i, \rho) \delta(1 - \sum_{i=1}^3 \eta_i)}{[\eta_1 z_1 + \eta_2 z_2 + \eta_3 z_3 + \rho - i0]}, \quad (11b)$$

$$\begin{aligned} \rho_0(\eta_i) &= \max_{z_i \in G_0} [-\eta_1 z_1 - \eta_2 z_2 - \eta_3 z_3] \\ &= \eta_1(m^2 + \mu^2) + \eta_2(m^2 + \mu^2) + 2\eta_3\mu^2 - 2 \min_{\zeta_i \in G_0} (m\mu\eta_1\zeta_1 \\ &\quad + m\mu\eta_2\zeta_2 + \mu^2\eta_3\zeta_3). \end{aligned} \quad (12)$$

The minimum with respect to ζ_i in (12) is attained in the region G_0 , where the ζ_i may be specified in the form $\zeta_i = \cos \theta_i$, and gives the following values for $\rho_0(\eta_i)$ ⁽⁶⁾:

$$\rho_0(\eta_i) = \Phi(\eta_i) \left[\mu^2 \left(\frac{1}{\eta_1} + \frac{1}{\eta_2} \right) + \frac{m^2}{\eta_3} \right], \quad \text{if } \mu \left| \frac{1}{\eta_1} - \frac{1}{\eta_2} \right| < \frac{m}{\eta_3} < \mu \left(\frac{1}{\eta_1} + \frac{1}{\eta_2} \right),$$

$$\rho_0(\eta_i) = \eta_1(m + \mu)^2 + \eta_2(m - \mu)^2 + 4\mu^2\eta_3, \quad \text{if } \frac{\mu}{\eta_2} > \frac{\mu}{\eta_1} + \frac{m}{\eta_3},$$

$$\rho_0(\eta_i) = \eta_1(m + \mu)^2 + \eta_2(m + \mu)^2, \quad \text{if } \frac{m}{\eta_3} > \mu \left(\frac{1}{\eta_1} + \frac{1}{\eta_2} \right),$$

$$\rho_0(\eta_i) = \eta_1(m - \mu)^2 + \eta_2(m + \mu)^2 + 4\mu^2\eta_3, \quad \text{if } \frac{\mu}{\eta_1} > \frac{\mu}{\eta_2} + \frac{m}{\eta_3};$$

$$\Phi(\eta_i) = \eta_1\eta_2 + \eta_2\eta_3 + \eta_1\eta_3.$$

In conclusion, the author expresses his deep gratitude to A. A. Logunov and A. N. Tavkhelidze for their interest in the work and for discussion.

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Received
12 XII 1964

References

1. L. D. Faddeev, *ZhETF*, **39**, 1459 (1960).
2. I. Sh. Vashakidze, R. M. Muradian, A. N. Tavkhelidze, G. A. Chilashvili, V. P. Shelest, JINR Preprint R-1662, Dubna, 1964; L. Rosenberg, *Phys. Rev.*, **131**, 874 (1963).
3. N. Nakanishi, *Suppl. Progr. Theor. Phys.*, No. 18 (1961).
4. A. A. Logunov, I. T. Todorov, N. A. Chernikov, JINR Preprint D-578, Dubna, 1960; *ZhETF*, **42**, 1285 (1962).
5. O. I. Zavialov, *DAN*, **151**, No. 3 (1963).
6. L. D. Landau, *ZhETF*, **37**, 62 (1959).
7. Liu I-ch' en, I. T. Todorov, *DAN*, **148**, No. 4 (1963).
8. T. T. Wu, *Phys. Rev.*, **123**, 678 (1961).

Note: Figure translations are in progress. See original paper for figures.

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