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Abstract

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THEORY OF ELASTICITY

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THE INTEGRAL-OPERATOR METHOD IN THE HEREDITARY THEORY OF CREEP

(Presented by Academician A. Yu. Ishlinskii, 23 VII 1964)

1. Properties of integral operators. The Volterra integral operator with kernel $\mathfrak{A}_\alpha(x_i; t-s)$ of the Yu. N. Rabotnov type ⁽¹⁾, acting on a certain function $\zeta(t)$, has the form

$$\mathfrak{A}_\alpha^*(x_i)\zeta(t) = \int_0^t \mathfrak{A}_\alpha(x_i; t-s)\zeta(s) ds = \int_0^t \sum_{n=0}^{\infty} \frac{x_i^n (t-s)^{n(1+\alpha)+\alpha}}{\Gamma[(n+1)(1+\alpha)]} \zeta(s) ds \quad (\alpha > -1). \quad (1)$$

The kernel of the power of the operator $\mathfrak{A}_\alpha^{*m_i}(x_i)$, where x_i is a parameter, has the form

$$\mathfrak{A}_{\alpha; m_i}(x_i; t-s) = \int_s^t \mathfrak{A}_\alpha(x_i; t-\tau) \mathfrak{A}_{\alpha; m_i-1}(x_i; \tau-s) d\tau \quad (m_i = 2, 3, \dots), \quad (2)$$

where $\mathfrak{A}_{\alpha; m_i-1}(x_i; t-\tau)$ are repeated kernels of the initial kernel $\mathfrak{A}_\alpha(x_i; t)$. The kernel of the product $\mathfrak{A}_\alpha^{*m_1}(x_1)\mathfrak{A}_\alpha^{*m_2}(x_2)$ is expressed through the kernels $\mathfrak{A}_{\alpha; m_1}(x_1; t-s)$ and $\mathfrak{A}_{\alpha; m_2}(x_2; t-s)$ as follows:

$$\int_s^t \mathfrak{A}_{\alpha; m_1}(x_1; t-\tau) \mathfrak{A}_{\alpha; m_2}(x_2; \tau-s) d\tau. \quad (3)$$

In the corresponding manner one constructs the kernel of the operator $\prod_{i=1}^n \mathfrak{A}_\alpha^{*m_i}(x_i)$. The operators $\mathfrak{A}_\alpha^*(x_i)$ possess the properties of commutativity, distributivity, and associativity. They also admit the operation of differentiation with respect to the parameter x_i when acting on any integrable function $\zeta(t)$.

For the effective solution of problems of the hereditary theory of creep by the integral-operator method, in particular by applying Volterra's principle ⁽¹⁾,

the aforementioned properties are insufficient, since they do not eliminate the necessity of finding repeated kernels and their compositions. To a certain extent these difficulties can be overcome by invoking two properties of the $\mathfrak{E}_\alpha^*(x_i)$ -operators established by Yu. N. Rabotnov ⁽¹⁾:

$$(x_1 - x_2)\mathfrak{E}_\alpha^*(x_1)\mathfrak{E}_\alpha^*(x_2) = \mathfrak{E}_\alpha^*(x_1) - \mathfrak{E}_\alpha^*(x_2) \quad (x_1 \neq x_2), \quad (4)$$

$$[1 - \chi\mathfrak{E}_\alpha^*(-\beta)]^{-1} = 1 + \chi\mathfrak{E}_\alpha^*(\chi - \beta), \quad (5)$$

as well as the property of the power of the operator

$$\mathfrak{E}_\alpha^{*m_i}(x_i) = \frac{1}{(m_i - 1)!} \frac{\partial^{m_i-1} \mathfrak{E}_\alpha^*(x_i)}{\partial x_i^{m_i-1}}, \quad (6)$$

recently established by the author ⁽²⁾ of the present communication.

Properties (4) and (5) are used in solving static problems of the linear theory of creep of homogeneous and isotropic bodies. Property (6), as shown in [2], makes it possible to solve efficiently dynamic problems of the linear theory of creep of homogeneous and isotropic bodies. This property of the operator degree can also be used in solving certain static problems of the nonlinear theory of creep of inhomogeneous bodies.

Using the method of mathematical induction and formulas (4) and (6), one can establish a new property of the $\mathfrak{E}_\alpha^*(x_i)$ -operators, which can be successfully applied in solving dynamic problems of the linear theory of creep of inhomogeneous and anisotropic bodies. The property in question is expressed by the formula

$$\prod_{i=1}^n \mathfrak{E}_\alpha^{*m_i}(x_i) = \frac{1}{\prod_{i=1}^n (m_i - 1)!} \frac{\partial^{m_1+m_2+\dots+m_n-n}}{\partial x_1^{m_1-1} \partial x_2^{m_2-1} \dots \partial x_n^{m_n-1}} \left[\sum_{i=1}^n \frac{\mathfrak{E}_\alpha^*(x_i)}{\prod_{k=1, k \neq i}^n (x_i - x_k)} \right] \quad (7)$$

$$(i \neq k; x_1 \neq x_2 \neq \dots \neq x_n).$$

Formula (7) makes it possible to express the product of different powers of the operators directly through derivatives with respect to the parameters x_i of the initial kernels $\mathfrak{E}_\alpha^*(x_i)$, and thus to avoid the need to perform operations more complicated than operations of types (2), (3).

To verify the validity of formula (7), let us first form the product of two operators $\mathfrak{E}_\alpha^{*m_1}(x_1)$ and $\mathfrak{E}_\alpha^{*m_2}(x_2)$, using formula (6), as well as the associative property of both differential operators and integral operators. We shall have

$$\mathcal{E}_\alpha^{*m_1}(x_1)\mathcal{E}_\alpha^{*m_2}(x_2) = \frac{1}{(m_1-1)!} \frac{\partial^{m_1-1}\mathcal{E}_\alpha^*(x_1)}{\partial x_1^{m_1-1}} \frac{1}{(m_2-1)!} \frac{\partial^{m_2-1}\mathcal{E}_\alpha^*(x_2)}{\partial x_2^{m_2-1}} = \frac{1}{(m_1-1)!(m_2-1)!} \frac{\partial^{m_2-1}}{\partial x_2^{m_2-1}} \left\{ \frac{\partial^{m_1}}{\partial x_1^{m_1}} \right\}$$

Let

$$\prod_{i=1}^{n-1} \mathcal{E}_\alpha^{*m_i}(x_i) = \frac{1}{\prod_{i=1}^{n-1} (m_i-1)!} \frac{\partial^{m_1+m_2+\dots+m_{n-1}-n+1}}{\partial x_1^{m_1-1} \partial x_2^{m_2-1} \dots \partial x_{n-1}^{m_{n-1}-1}} \left[\prod_{i=1}^{n-1} \mathcal{E}_\alpha^*(x_i) \right].$$

Then

$$\prod_{i=1}^n \mathcal{E}_\alpha^{*m_i}(x_i) = \mathcal{E}_\alpha^{*m_n}(x_n) \prod_{i=1}^{n-1} \mathcal{E}_\alpha^{*m_i}(x_i) = \frac{1}{\prod_{i=1}^n (m_i-1)!} \frac{\partial^{m_1+m_2+\dots+m_{n-1}-n+1}}{\partial x_1^{m_1-1} \partial x_2^{m_2-1} \dots \partial x_{n-1}^{m_{n-1}-1}} \left\{ \frac{\partial^{m_n-1}}{\partial x_n^{m_n-1}} \left[\mathcal{E}_\alpha^*(x_n) \prod_{i=1}^{n-1} \mathcal{E}_\alpha^*(x_i) \right] \right\}$$

That

$$\prod_{i=1}^n \mathcal{E}_a^*(x_i) = \sum_{i=1}^n \left[\prod_{k=1}^n (x_i - x_k) \right]^{-1} \mathcal{E}_a^*(x_i), \quad (9)$$

can also be verified by the method of mathematical induction. Indeed, for $n = 2$ relation (9) coincides with formula (4). Let

$$\prod_{i=1}^{n-1} \mathcal{E}_a^*(x_i) = \sum_{i=1}^{n-1} \left[\prod_{k=1}^n (x_i - x_k) \right]^{-1} \mathcal{E}_a^*(x_i).$$

Then

$$\begin{aligned} \prod_{i=1}^n \mathcal{E}_a^*(x_i) &= \mathcal{E}_a^*(x_n) \prod_{i=1}^{n-1} \mathcal{E}_a^*(x_i) \\ &= \sum_{i=1}^{n-1} \left[\prod_{l=1}^n (x_i - x_l) \right]^{-1} [\mathcal{E}_a^*(x_i) - \mathcal{E}_a^*(x_n)] \\ &= \sum_{i=1}^n \left[\prod_{k=1}^n (x_i - x_k) \right]^{-1} \mathcal{E}_a^*(x_i), \end{aligned}$$

for

$$\sum_{i=1}^{n-1} \left[\prod_{k=1}^n (x_i - x_k) \right]^{-1} = - \left[\prod_{k=1}^{n-1} (x_n - x_k) \right]^{-1}.$$

Thus, the validity of formula (9) and, consequently, property (7), is proved.

§ 2. Vibrations of an orthotropic plate. Consider the free vibrations of an orthotropic elastic-hereditary plate with four supported sides, which at $t = 0$ has received the initial deflection $w(x, y; 0) = w_0(x, y)$ ($a \leq x, y \leq a$). We take the initial velocity of deflection to be zero. Using Volterra's principle, extended in paper ⁽²⁾ to dynamic problems, we obtain the deflection of the plate

$$w(x, y; t) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \alpha_{n_1 n_2} \cos \bar{p}_{n_1 n_2} t \sin \frac{n_1 \pi x}{a} \sin \frac{n_2 \pi y}{a}, \quad (10)$$

where

$$\alpha_{n_1 n_2} = \frac{4}{a^2} \int_0^a \int_0^a w_0(x, y) \sin \frac{n_1 \pi x}{a} \sin \frac{n_2 \pi y}{a}.$$

Relation (10) is the result of replacing, in the known solution of the corresponding elastic problem ⁽³⁾, the frequency $p_{n_1 n_2}$ by the frequency operator

$$\bar{p}_{n_1 n_2} = \frac{\pi^2}{a^2} \sqrt{\frac{g}{h\gamma} (\bar{D}_1 n_1^4 + 2\bar{D}_3 n_1^2 n_2^2 + \bar{D}_2 n_2^4)}, \quad (11)$$

where γ is the specific weight of the material, h is the thickness of the plate;

$$\bar{D}_i = D_{i0} [1 - \chi_i \mathcal{E}_a^*(-\beta_i)] \quad (i = 1, 2, 3) \quad (12)$$

are stiffness operators; D_{i0} are the instantaneous stiffnesses, and χ_i and β_i are rheological characteristics of the plate. From (11), taking (12) into account, we obtain

$$\bar{p}_{n_1 n_2} = p_{n_1 n_2; 0} \left[1 - \sum_{i=1}^3 k_i \mathcal{E}_a^*(-\beta_i) \right]^{1/2}, \quad (13)$$

where

$$p_{n_1 n_2; 0} = \frac{\pi^2}{a^2} \sqrt{\frac{g T_{n_1 n_2}}{h\gamma}}, \quad T_{n_1 n_2} = D_{10} n_1^4 + 2D_{30} n_1^2 n_2^2 + D_{20} n_2^4,$$

$$k_1 = \frac{D_{10}n_1^4\chi_1}{T_{n_1n_2}}, \quad k_2 = \frac{D_{20}n_2^4\chi_2}{T_{n_1n_2}}, \quad k_3 = \frac{2D_{30}n_1^2n_2^2\chi_3}{T_{n_1n_2}}.$$

To decipher the operator expression $\cos \bar{p}_{n_1n_2} t$ appearing in (10), we carry out the following expansions, taking into account the expression for the operator $\bar{p}_{n_1n_2}$ according to (13):

$$\begin{aligned} \cos \bar{p}_{n_1n_2} t &= \sum_{m=0}^{\infty} (-1)^m \frac{p_{n_1n_2;0}^{2m}}{(2m)!} \left[1 - \sum_{i=1}^3 k_i \mathcal{E}_\alpha^i(-\beta_i) \right]^m t^{2m} \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{p_{n_1n_2;0}^{2m}}{(2m)!} \left[1 - \sum_{\nu=1}^m (-1)^{\nu-1} \binom{m}{\nu} G_\nu \right] t^{2m}, \end{aligned} \quad (14)$$

where

$$G_\nu = \sum_{s=0}^{\nu} \binom{\nu}{s} \sum_{j=0}^{\nu-s} \binom{\nu-s}{j} [k_1 \mathcal{E}_\alpha^*(-\beta_1)]^j [k_2 \mathcal{E}_\alpha^*(-\beta_2)]^{\nu-s-j} [k_3 \mathcal{E}_\alpha^*(-\beta_3)]^s.$$

Applying formula (7) in representation (14) for $n = 3$ and $m_1 = j$, $m_2 = \nu - s - j$, $m_3 = s$, and $x_i = -\beta_i$ ($i = 1, 2, 3$), we obtain

$$\cos \bar{p}_{n_1n_2} t = \cos p_{n_1n_2;0} t - \sum_{m=0}^{\infty} (-1)^m \frac{p_{n_1n_2;0}^{2m}}{(2m)!} \sum_{\nu=1}^m \binom{m}{\nu} \mathcal{L}_\nu(t), \quad (15)$$

where

$$\mathcal{L}_\nu(t) = \sum_{s=0}^{\nu} (-1)^s \binom{\nu}{s} \sum_{j=0}^{\nu-s} \binom{\nu-s}{j} \frac{k_1^j k_2^{\nu-s-j} k_3^s}{\prod_{i=1}^3 (m_i - 1)!} \frac{\partial^{\nu-3} \sum_{i=1}^3 \frac{\mathcal{E}_\alpha^*(-\beta_i) t^{2m}}{\prod_{\substack{k=1 \\ k \neq i}}^3 (\beta_i - \beta_k)}}{\partial \beta_1^{j-1} \partial \beta_2^{\nu-s-j-1} \partial \beta_3^{s-1}}. \quad (16)$$

The computations in formula (16), after carrying out the operation of differentiation with respect to the parameters β_i ($i = 1, 2, 3$), are performed by means of the formula

$$\frac{\partial^{\lambda-1}}{\partial \beta^{\lambda-1}} \mathcal{E}_\alpha^*(-\beta) t^r = \int_0^t \frac{\partial^{\lambda-1} \mathcal{E}_\alpha(-\beta; t - \tau)}{\partial \beta^{\lambda-1}} \tau^r d\tau = \sum_{q=\lambda-1}^{\infty} \frac{r! q(q-1) \dots (q-\lambda+2)}{\Gamma[(q+1)(1+\alpha) + r + 1]} (-\beta)^{q-\lambda+1} t^{(q+1)(1+\alpha)+r}$$

for $r = 2m$, $\beta = \beta_1; \beta_2; \beta_3$ and $\lambda = j; \nu - s - j; s$.

The series appearing in representation (15) converges absolutely and uniformly on every finite interval of variation of the time t .

It follows from representation (15) that the operator $\cos \bar{p}_{n_1 n_2} t$ is expressed as the difference of a periodic component, corresponding to the ideally elastic state, and an aperiodic component taking into account the hereditary orthotropic properties of the material.

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CITED LITERATURE

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³ S. G. Lekhnitskii, *Anisotropic Plates*, Moscow, 1957.

Note: Figure translations are in progress. See original paper for figures.

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