

FREE (T) -SUMS OF MULTIOPERATOR FIELDS

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Abstract

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MATHEMATICS

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FREE T -SUMS OF MULTIOPERATOR FIELDS

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The theory, constructed by L. A. Skorniyakov ^(2,3), of free T -extensions of linear algebras is transferred in the present note to a broader class of multioperator algebras introduced by A. G. Kurosh ⁽¹⁾. Theorem 2 gives a positive solution to the question of the existence of isomorphic continuations for any two free T -decompositions of a multioperator, in particular nonassociative, field. For a nonassociative field this question was not considered in ⁽³⁾, although it is natural after Theorem 2 and Theorem § 5 of ⁽³⁾. Theorem 1 generalizes the analogous theorem of ⁽³⁾. The incorrectness of the original proof of this theorem (in ⁽³⁾) is noted.

Below, a nonempty system of operations $\Omega = \{\omega, \dots\}$, $n(\omega) \geq 2$, and a field P are assumed fixed.

§ 1. An Ω -algebra A over the field P ⁽¹⁾ is called an Ω -algebra with divisions (a multioperator field, or Ω -field) if, for each $\omega \in \Omega$, every equation of the form

$$a_1 \dots a_{i-1} x a_{i+1} \dots a_n \omega = c, \quad 0 \neq a_j, \quad c \in A, \quad (*)$$

is solvable (uniquely solvable) in A for $i = 1, 2, \dots, n = n(\omega)$.

A single-valued mapping θ of an Ω -algebra A into an Ω -algebra K augmented formally by the symbol ∞ is called a T -homomorphism ⁽²⁾ if $\theta^{-1}(0)$ is nonempty and the following conditions are satisfied:

1. If $a_i \in A$, $\xi_j \in P$, $\theta(a_i) \neq \infty$, $i = 1, 2, \dots, n$, then

$$\theta(\xi_1 a_1 + \xi_2 a_2) = \xi_1 \theta(a_1) + \xi_2 \theta(a_2), \quad \theta(a_1 \dots a_n \omega) = \theta(a_1) \dots \theta(a_n) \omega, \quad \omega \in \Omega, \quad n = n(\omega).$$

2. If $\theta(a_i) \neq 0$, $i = 1, 2, \dots, n$, and for at least one $1 \leq i_0 \leq n$

$$\theta(a_{i_0}) = \infty,$$

then

$$\theta(a_1 \dots a_{i_0} \dots a_n \omega) = \infty.$$

It is immediately verified that the T -homomorphic image of an Ω -algebra with divisions is an Ω -algebra with divisions. The successive performance of two T -homomorphisms is again a T -homomorphism, if, by definition, one sets $\theta(\infty) = \infty$.

§ 2. Let a T -homomorphism θ and an equation $(*)$, not solvable in the Ω -algebra A , be fixed. A basis $\Sigma = \Phi \cup \Psi$ of the additive vector space A^+ of the Ω -algebra A will be called $(\theta, *)$ -regular if: 1) Φ is a basis of the subspace $E(\Phi) = \{a \in A^+ \mid \theta(a) \neq \infty\}$; 2) for all $a \in E(\Psi) \setminus 0$, $\theta(a) = \infty$; 3) the subspace

$$L = E(\{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n\}),$$

where the a_j are the terms of equation $(*)$, has the basis set

$$\Sigma \cap L = \Sigma'.$$

In the basis S , constructed over the set $\Sigma \cup \lambda$ ⁽²⁾, of the free sum

$$A_\lambda = A * F(\lambda)$$

of the Ω -algebra A and the free Ω -algebra $F(\lambda)$ with one free generator λ , we fix

$$b_1 = a'_1 \dots a'_{i-1} \lambda a'_{i+1} \dots a'_n \omega, \quad a'_j \in \Sigma',$$

$$E(\{a'_1, \dots, a'_{i-1}, a'_{i+1}, \dots, a'_n\}) = L.$$

The word b_1 occurs as a summand in the S -canonical representation of the element

$$p = a_1 \dots a_{i-1} \lambda a_{i+1} \dots a_n \omega - c$$

corresponding to equation $(*)$. Then, under the natural epimorphism

$$A_\lambda \rightarrow B = A_\lambda / I,$$

where I is the ideal of the Ω -algebra A_λ generated by the element p , the set

$$S^{b_1} = \{x \in S \mid l_{b_1}(x) = 0\}$$

of elements of b_1 -length zero ⁽²⁾ is mapped one-to-one onto a basis of the vector space B^+ of the Ω -algebra B . In the Ω -algebra $B \supset A$, equation $(*)$ is solvable (see ⁽²⁾). Using

basis S^{b_1} , one can carry over to the case of multioperator algebras the proof of Lemma 6 of ⁽²⁾, stating that every T -homomorphism of an Ω -algebra A into an Ω -algebra K with divisions can be extended to the obtained extension B . Hence, as in ⁽²⁾, we obtain:

Proposition 1. Every Ω -algebra A over a field P possesses a free T -extension $\mathfrak{A} \supset A$ in the sense of ⁽²⁾, uniquely determined up to an isomorphism identical on A . If, moreover, for any $0 \neq x_j \in A$, $\omega \in \Omega$, one has $x_1 \dots x_n \omega \neq 0$, then $\mathfrak{A}$ is an Ω -field.

Corollary 1. Every Ω -algebra can be embedded in an Ω -algebra with divisions.

This corollary generalizes B. H. Neumann's theorem ⁽⁴⁾, proved for linear algebras.

Proposition 1 makes it possible to define the concepts (see ^(2,3)): free Ω -field, free T -sum of Ω -fields. The following propositions are true and are immediate generalizations of theorems from ^(2,3).

Proposition 2. Every Ω -field is a T -homomorphic image of some free Ω -field.

Proposition 3. An Ω -subfield $\mathfrak{B}$ of the free T -sum $\sum_T \mathfrak{A}_\sigma * \mathfrak{F}$ is decomposable into the free T -sum

$$\mathfrak{B} = \sum_T (\mathfrak{B} \cap \mathfrak{A}_\sigma) * \mathfrak{F}_1,$$

where $\mathfrak{F}$ and $\mathfrak{F}_1$ are free Ω -fields.

For free T -sums of multioperator fields, besides properties analogous to I–V in ⁽³⁾, the following also hold:

A1 (validity of Mal'cev's postulate). If in each of the Ω -fields $\mathfrak{A}_\alpha$ an Ω -subfield $\mathfrak{B}_\alpha$ is taken and $\mathfrak{A} = \sum_T \mathfrak{A}_\alpha$, then the Ω -subfield $\mathfrak{B}$ generated by all $\mathfrak{B}_\alpha$ is their free T -sum and $\mathfrak{B} \cap \mathfrak{A}_\alpha = \mathfrak{B}_\alpha$.

A2. If M is a finite subset of the Ω -field $\mathfrak{A} = \sum_T \mathfrak{A}_\alpha$, then there exists a finite set of indices $\alpha_1, \dots, \alpha_n$ such that

$$M \subset \sum_{i=1}^{nT} \mathfrak{C}_{\alpha_i},$$

where $\mathfrak{C}_{\alpha_i}$ is an Ω -subfield of the Ω -field $\mathfrak{A}_{\alpha_i}$. Each $\mathfrak{C}_{\alpha_i}$ has, as an Ω -field, a finite number of generators.

§ 3. Let, for the Ω -field $\mathfrak{A}$, there be a free T -decomposition:

$$\mathfrak{A} = \sum_T \mathfrak{A}_\sigma.$$

Then, by the definition of the free T -sum,

$$\Sigma^* \mathfrak{A}_\sigma = A_1 \subset \dots \subset A_\beta \subset \dots \subset A_\theta = \mathfrak{A}.$$

In this transfinite sequence of extensions, for every $0 \neq x \in \mathfrak{A}$ there is uniquely determined a height $h(x)$ —such an indefinite ordinal number $\alpha \geq 1$ that $x \in$

$A_\alpha \setminus A_{\alpha-1}$ (here A_0 is to be understood as the empty set), and a degree $s_\alpha(x)$ ⁽²⁾.

The Ω -subfield generated by a set $Z \subset \mathfrak{A}$ will be denoted by $T(Z)$. Suppose that, as an Ω -field, $\mathfrak{A}$ is generated by some finite subset of it. An ordered set $X = \{x_i \in \mathfrak{A} \mid x_i \neq 0, i = 1, \dots, m\}$ will be called a **finite system of generators** (f.s.g.) of the Ω -field $\mathfrak{A}$, if $T(X) = \mathfrak{A}$ and $h(x_i) \leq h(x_{i+1})$, $i = 1, 2, \dots, m-1$. If the subset $X_\alpha = X \cap (A_\alpha \setminus A_{\alpha-1})$ of the f.s.g. X is nonempty, then by $s_\alpha(X)$ we shall denote the sum $\sum_i s_\alpha(x_i)$, $x_i \in X_\alpha$.

On the set $\mathfrak{P}$ of all f.s.g. of the Ω -field $\mathfrak{A}$ we define a quasi-ordering. Let $X = \{x_1, \dots, x_m\}$, $Y = \{y_1, \dots, y_n\} \in \mathfrak{P}$. By definition, we set $X \leq Y$ if one of the following conditions is satisfied: 1) $m < n$, 2) $m = n$, $h(x_i) \leq h(y_i)$, $i = 1, 2, \dots, n$, and at least one of the inequalities is strict; 3) $m = n$, $h(x_i) = h(y_i)$, $s_{\alpha_i}(X) \leq s_{\alpha_i}(Y)$ for all $\alpha_i = h(x_i)$, $i = 1, 2, \dots, n$. It is easily verified that the relation thus defined is reflexive and transitive and therefore is a partial ordering on the set $\mathfrak{P}$ of classes of equivalent f.s.g.: $X \sim Y$ if $m = n$, $h(x_i) = h(y_i)$, $s_{\alpha_i}(X) = s_{\alpha_i}(Y)$ for all $\alpha_i = h(x_i)$, $i = 1, 2, \dots, n$. Since the minimality condition holds for $\mathfrak{P}$, we thus arrive at the definition of **minimal** f.s.g. (m.f.s.g.) of Ω -fields $\mathfrak{A}$. It is clear that

the number of elements in any two m.k.s.o. is the same and does not depend on the T -decomposition under consideration.

Denote by $n(X)$ and $r(X)$, respectively, the number of elements of a finite set X of the Ω -group $\mathfrak{A} = \Sigma^T \mathfrak{A}_\sigma$ and of its subset $X \setminus (\bigcup_\sigma \mathfrak{A}_\sigma)$.

Let, further,

$$\bar{A}_0 = \{x \in A_1 \mid s_1(x) = 0\},$$

where $A_1 = \Sigma^* \mathfrak{A}_\sigma$; $X_0 = X \cap \bar{A}_0$. If X is a k.s.o. of the Ω -group $\mathfrak{A}$ and there exists a finite set $\bar{X}_0 \subset \bar{A}_0$ such that

$$T(\bar{X}_0) = T(X_0), \quad r(\bar{X}_0) < r(X_0), \quad n(\bar{X}_0) \leq n(X_0) + 1,$$

then the passage to the k.s.o. $X' = (X \setminus X_0) \cup \bar{X}_0$ will be called an r -reduction. Any k.s.o. X , in no more than $r(X_0)$ steps, is brought to an r -irreducible form that admits no r -reductions.

Let Y be an m.k.s.o. of the Ω -group $\mathfrak{A}$, and let X be the r -irreducible k.s.o. obtained from Y . Denote it by $X = \pi(Y)$ and call it a regular k.s.o. of the Ω -group $\mathfrak{A}$.

As an analogue of Grushko's theorem, the following has been obtained.

Proposition 4. *The elements of a regular k.s.o. $X = \pi(Y)$ belong to separate T -summands.*

The proof of this proposition is based on a specialization of the methods, developed in papers ^(1, 3, 5), for constructing bases in the Ω -subalgebra of a free sum of Ω -algebras.

Corollary 2. *An Ω -group with n generators decomposes into a free T -sum of no more than $2n$ T -indecomposable finitely generated Ω -subgroups.*

Corollary 3. *A free Ω -group with n free generators is not decomposable into a free T -sum containing more than n summands.*

Hence, using property A2, we arrive at theorem *:

Theorem 1. *The cardinality of the set of free generators of a free Ω -group is an invariant of this Ω -group.*

Theorem 2. *Any two free T -decompositions of an Ω -group have isomorphic refinements.*

To prove Theorem 2, in view of Proposition 3 and properties III and IV for free T -sums (see ⁽³⁾), it suffices to establish an isomorphism of the free Ω -groups $\mathfrak{F}_1$ and $\mathfrak{F}_2$ for any two free T -decompositions:

$$\mathfrak{C} *_T \mathfrak{F}_1 = \mathfrak{C} *_T \mathfrak{F}_2,$$

where $\mathfrak{C}$ is an arbitrary Ω -group.

Let M_1 and M_2 be some fixed systems of free generators of the free Ω -groups $\mathfrak{F}_1$ and $\mathfrak{F}_2$, and suppose

$$\text{card } M_1 < \text{card } M_2, \quad (**)$$

and M_2 is an infinite set. Then, using A2, by the usual argument we arrive at a contradiction. If $\mathfrak{C}$ is generated, as an Ω -group, by a finite set and if M_2 is finite, then (**) leads to a contradiction with Corollary 2. But when M_1, M_2 are finite, by virtue of property A2 one may restrict oneself only to the case considered, that of a finitely generated Ω -group $\mathfrak{C}$. Theorem 2 is proved.

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* In paper ⁽³⁾ this is Theorem § 5, formulated for a free nonassociative ring. The proof given there is unconvincing, since it contains an incorrect reference (p. 309) to paper ⁽⁵⁾. As for Theorem 5 of ⁽³⁾, which is also affected by the noted incorrectness, the question of the validity of this theorem has not been clarified. Corollary 2 of the present note contains a weaker assertion.

Note: Figure translations are in progress. See original paper for figures.

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