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Abstract

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MATHEMATICS

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ON THE DEPENDENCE ON THE BOUNDARY OF SOLUTIONS OF SOME MIXED PROBLEMS FOR THE EQUATIONS OF SMALL OSCILLATIONS OF A ROTATING FLUID

(Presented by Academician S. L. Sobolev on 22 July 1965)

Consider the system of equations describing small oscillations of a rotating fluid:

$$\frac{\partial u}{\partial t} = v - \frac{\partial p}{\partial x}; \quad \frac{\partial v}{\partial t} = -u - \frac{\partial p}{\partial y}; \quad \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z}; \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (1)$$

S. L. Sobolev proved the existence and uniqueness of the solution of (1) satisfying the initial conditions

$$u|_{t=0} = u_0(x, y, z); \quad v|_{t=0} = v_0(x, y, z); \quad w|_{t=0} = w_0(x, y, z) \quad (2)$$

$$(\mathbf{U}|_{t=0} = \mathbf{U}_0(x, y, z))$$

and one of the boundary conditions:

$$p|_{\Gamma} = 0; \quad (3)$$

$$u \cos(n, x) + v \cos(n, y) + w \cos(n, z)|_{\Gamma} = 0, \quad (4)$$

where Γ is the boundary of the domain Ω in the space x, y, z . In the case of the boundary condition (4), the vector $\mathbf{U} = (u, v, w)$ is determined uniquely, and p up to an addend depending only on t . The behavior of the solutions of problem (1), (2), (3) in the plane case (when Ω is an infinite cylinder with generator parallel to the y -axis; $\partial p / \partial y \equiv 0$) was investigated in ^(3,4,7). In this case, in ⁽³⁾,

examples were constructed of domains arbitrarily close to a circle in which there exist solutions that are not almost periodic, although in the case of the circle all solutions are almost periodic. If $\mathbf{U}, p$ satisfy (1)–(3), then p is a solution of the problem

$$\frac{\partial^2}{\partial t^2} \Delta p + \frac{\partial^2 p}{\partial z^2} = 0; \quad (5)$$

$$p|_{t=0} = p_0(x, y, z); \quad \left. \frac{\partial p}{\partial t} \right|_{t=0} = p_1(x, y, z); \quad (6)$$

$$p|_{\Gamma} = 0, \quad (7)$$

where p_0, p_1 are determined uniquely by $\mathbf{U}_0$. In the case of condition (4), (7) is replaced by the following boundary condition:

$$\frac{\partial^2}{\partial t^2} \frac{dp}{dn} + \frac{\partial}{\partial t} \left[\frac{\partial p}{\partial y} \cos(n, x) - \frac{\partial p}{\partial x} \cos(n, y) \right] + \frac{\partial p}{\partial z} \cos(n, z) \Big|_{\Gamma} = 0. \quad (8)$$

In the plane case, condition (8) has the form

$$\frac{\partial^2}{\partial t^2} \frac{dp}{dn} + \frac{\partial p}{\partial z} \cos(n, z) \Big|_{\gamma} = 0, \quad (9)$$

where γ is the boundary of the projection D of the cylinder Ω onto the plane $y = 0$. In (7,8) an infinite-dimensional subspace of the space $\dot{W}_2^1(D)$ was constructed, a basis in which is formed by functions of the form

$$f(x + \mu_{k,1}^N(x, z)z) + g(x + \mu_{k,2}^N(x, z)z) \quad (k = 0, \dots, N-1; N = 2, 3, \dots),$$

and the solutions of the mixed problem (1)–(3) in the planar case, for initial data from this subspace, are linear combinations of functions of the form

$$\int_{\mu_{k,1}^N(x, z)}^{\mu_{k,2}^N(x, z)} f(\alpha) e^{i\lambda(\alpha)t} d\alpha \quad (10)$$

and of almost periodic functions of t .

The functions $\mu_{k,i}^N$ depend continuously on the domain. S. L. Sobolev [2] established continuous dependence on the domain, on a finite time interval, of the solutions of problem (1)–(3), by estimating the norm of the difference between

the solution (U, p) in the domain Ω and the solution constructed in the domain Ω_1 , containing Ω , which satisfies the initial condition

$$U_1|_{t=0} = \begin{cases} U|_{t=0}, & \text{if } (x, y, z) \in \Omega, \\ 0, & \text{if } (x, y, z) \in \Omega_1 - \Omega. \end{cases}$$

It follows from this estimate that the indicated difference in Ω is small, provided only that the boundaries of the domains Ω and Ω_1 are sufficiently close; moreover, the order of convergence of this difference to zero as the boundaries approach one another depends on U_0 .

Consider the planar case of problem (1)–(3). Let D_ε be a one-parameter family of simply connected domains in the x, z -plane with boundaries γ_ε . For simplicity, consider the case where γ_0 is a circle, $\gamma_\varepsilon \rightarrow \gamma_0$. Suppose γ_ε is given by the equations $x = x_\varepsilon(t)$, $y = y_\varepsilon(t)$. Require that

$$\|x_0 - x_\varepsilon\|_{C_{k+\alpha}} \xrightarrow{\varepsilon \rightarrow 0} 0, \quad \|y_0 - y_\varepsilon\|_{C_{k+\alpha}} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

where k is some natural number. Map the domains D_ε conformally onto D_0 by means of the function

$$f_\varepsilon(\zeta) = \zeta \exp\{u_\varepsilon + iv_\varepsilon\},$$

where $\zeta = x + iz$, $\Delta u_\varepsilon = 0$,

$$u_\varepsilon|_{\gamma_\varepsilon} = -\ln \sqrt{x^2 + y^2}|_{\gamma_\varepsilon}, \quad v_\varepsilon = \int_{(0,0)}^{(x,z)} \left(-\frac{\partial u_\varepsilon}{\partial x} dz + \frac{\partial u_\varepsilon}{\partial z} d\xi \right).$$

By virtue of our assumptions, using the estimates from [6], we obtain

$$\|u_\varepsilon\|_{C_{k+\alpha}(D_\varepsilon)} \leq K\delta(\varepsilon); \quad \|v_\varepsilon\|_{C_{k+\alpha}(D_\varepsilon)} \leq K\delta(\varepsilon). \quad (11)$$

In (11), K can be chosen independent of ε , if

$$\|x_\varepsilon\|_{C_{k+1}} \leq \widetilde{K}, \quad \|y_\varepsilon\|_{C_{k+1}} \leq \widetilde{K},$$

where $\widetilde{K}$ does not depend on ε .

In the coordinates

$$\xi = \operatorname{Re} f_\varepsilon(\zeta) = \varphi_\varepsilon(x, z); \quad \eta = \operatorname{Im} f_\varepsilon(\zeta) = \psi_\varepsilon(x, z) \quad (12)$$

equation (5) has the form

$$\frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 p}{\partial \xi^2} + \frac{\partial^2 p}{\partial \eta^2} \right) + \frac{1}{I_\varepsilon} L_1^\varepsilon p = 0, \quad (13)$$

where

$$I_\varepsilon = (\partial\varphi_\varepsilon/\partial x)^2 + (\partial\varphi_\varepsilon/\partial z)^2, \quad (14)$$

and $L_1^\varepsilon p$ is the expression for $\partial^2 p/\partial z^2$ in the new coordinates.

Introduce the operator Δ^{-1} : $\Delta^{-1}(\partial^2 p/\partial \xi^2 + \partial^2 p/\partial \eta^2) = p$, if $p|_{\gamma_0} = 0$. Then (13), under condition (7), can be written in the form

$$\frac{\partial^2 p}{\partial t^2} = -\Delta^{-1} \left(\frac{1}{I_\varepsilon} L_1^\varepsilon p \right) = A_\varepsilon p. \quad (15)$$

For any ε , the operator A_ε is a self-adjoint bounded operator in the space $\mathring{W}_2^2(D_0)$ with scalar product

$$[u, v] = \iint_{D_0} \left(\frac{\partial u}{\partial \xi} \frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial v}{\partial \eta} \right) d\xi d\eta.$$

It is also obvious that $-[p, p] \leq [A_\varepsilon p, p] \leq 0$.

Theorem 1. Suppose

$$\|x_0(t) - x_\varepsilon(t)\|_{C_{1+\alpha}} \xrightarrow{\varepsilon \rightarrow 0} 0; \quad \|x_\varepsilon\|_{C_2} \leq K_1; \quad \|y_0(t) - y_\varepsilon(t)\|_{C_{1+\alpha}} \xrightarrow{\varepsilon \rightarrow 0} 0; \quad \|y_\varepsilon\|_{C_2} \leq K_1.$$

Then

$$\|A_\varepsilon - A_0\| \xrightarrow{\varepsilon \rightarrow 0} 0,$$

if p_ε and p_0 are solutions of equation (15), respectively for arbitrary ε and $\varepsilon = 0$, constructed from the same initial data; then for any finite T , $-T \leq t \leq T$, the estimate holds

$$\|p_\varepsilon - p_0\|_{\mathring{W}_2^2(D_0)} \leq C(\varepsilon, T) \left\{ \|p_0|_{t=0}\|_{\mathring{W}_2^2(D_0)} + \left\| \frac{\partial p}{\partial t} \right|_{t=0} \right\}_{\mathring{W}_2^2(D_0)},$$

where $C(\varepsilon, T) \xrightarrow{\varepsilon \rightarrow 0} 0$.

The proof is obvious if one uses the equality

$$[(A_\varepsilon - A_0)p, p] = \iint_{D_0} \left[\left(\frac{\partial p}{\partial \eta} \right)^2 - \left(\frac{\partial p}{\partial \xi} \frac{\partial \varphi_\varepsilon}{\partial z} + \frac{\partial p}{\partial \eta} \frac{\partial \psi_\varepsilon}{\partial z} \right)^2 I_\varepsilon^{-1} \right] d\xi d\eta$$

and represents the solutions in the form of a power series in t .

Theorem 2. Suppose

$$\|x_\varepsilon\|_{C_3} \leq K_2; \quad \|y_\varepsilon\|_{C_3} \leq K_2;$$

$$\|x_\varepsilon(t) - x_0(t)\|_{C_{2+\alpha}} \xrightarrow{\varepsilon \rightarrow 0} 0; \quad \|y_\varepsilon(t) - y_0(t)\|_{C_{2+\alpha}} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Suppose $p_0(t)$ is a solution of equation (13) for $\varepsilon = 0$, satisfying the boundary condition (9). Then there exists a solution $p_\varepsilon(t)$ of equation (13), satisfying the conditions

$$p_\varepsilon(0) = p_0(0); \quad \left. \frac{\partial p_\varepsilon}{\partial t} \right|_{t=0} = \left. \frac{\partial p_0}{\partial t} \right|_{t=0}$$

and the boundary condition obtained from (9) by replacing (12) by

$$\frac{\partial^2}{\partial t^2} M_1^\varepsilon p_\varepsilon + M_2^\varepsilon p_\varepsilon \Big|_{\gamma_0} = 0 \quad (16)$$

and such that

$$\|p_\varepsilon(t) - p_0(t)\|_{C_{2+\alpha}(D_0)} \leq C_1(\varepsilon, T) \left\{ \|p_0(0)\|_{C_{2+\alpha}(D_0)} + \left\| \left. \frac{\partial p_0}{\partial t} \right|_{t=0} \right\|_{C_{2+\alpha}(D_0)} \right\}$$

for $-T \leq t \leq T$, where $C_1(\varepsilon, T) \xrightarrow{\varepsilon \rightarrow 0} 0$.

A solution of equation (13) satisfying the conditions specified in the theorem can be represented in the form of the series

$$p_\varepsilon = \sum_0^\infty p_i^\varepsilon(\xi, \eta) t^i,$$

where the functions p_i^ε for $i > 2$ are orthogonal functions, identically equal to one. It is not difficult to see that in this case p_i^ε are determined uniquely from the equation and boundary conditions, and for them an estimate is valid which guarantees uniform convergence of this series in $C_{2+\alpha}(D_0)$ on every finite time interval. Thus, to prove the theorem it suffices to compare the partial sum of the series for $p_0(t)$ with the corresponding partial sum for $p_\varepsilon(t)$, and the inequality appearing in the statement of Theorem 2 is obtained after applying the a priori estimates from (6).

Let now Ω_ε be a one-parameter family of domains in the space x, y, z , star-shaped with respect to one and the same point x_0, y_0, z_0 . Let Γ_ε be given by the equations $\zeta = \zeta_\varepsilon(\varphi, \theta)$, where (φ, θ, ζ) are the spherical coordinates of the point x, y, z . If $\|\zeta_\varepsilon\|_{C_3} \leq K_6$, K_6 is a constant independent of ε , and $\|\zeta_\varepsilon - \zeta_0\|_{C_{2+\alpha}} \rightarrow 0$ ($\varepsilon \rightarrow 0$), then Ω_ε is mapped onto Ω_0 by the mapping

$$\xi = \xi_\varepsilon(x, y, z), \quad \eta = \eta_\varepsilon(x, y, z), \quad \zeta = \zeta_\varepsilon(x, y, z),$$

where

$$\|\xi_\varepsilon - x\|_{C_{2+\alpha}}, \quad \|\eta_\varepsilon - y\|_{C_{2+\alpha}}, \quad \|\zeta_\varepsilon - z\|_{C_{2+\alpha}}$$

are small together with ε . The equation

(1) is then written in the form

$$\frac{\partial^2}{\partial t^2} N_\varepsilon p_\varepsilon + E_\varepsilon p_\varepsilon = 0,$$

and condition (4) in the form

$$\frac{\partial^2}{\partial t^2} \Phi_1^\varepsilon p_\varepsilon + \frac{\partial}{\partial t} \Phi_2^\varepsilon p_\varepsilon + \Phi_3^\varepsilon p_\varepsilon \big|_{\Gamma_0} = 0.$$

Analogously to the two-dimensional case, it is proved that, under the restrictions imposed,

$$\begin{aligned} & \|p_\varepsilon(\xi, \eta, \zeta, t) - p_0(\xi, \eta, \zeta, t)\|_{C_{2+\alpha}(\Omega_0)} \leq \\ & \leq K(T, \varepsilon) \left\{ \|p_0|_{t=0}\|_{C_{2+\alpha}(\Omega_0)} + \left\| \frac{\partial p_0}{\partial t} \bigg|_{t=0} \right\|_{C_{2+\alpha}(\Omega_0)} \right\}, \end{aligned}$$

where $p_\varepsilon|_{t=0} = p_0|_{t=0}$, $\partial p_\varepsilon / \partial t|_{t=0} = \partial p_0 / \partial t|_{t=0}$; $-T \leq t \leq T$, and p_ε is a solution of problem (1), (2), (3), or the correspondingly chosen solution of problem (1), (2), (4).

Thus it has been established that (U, p) , the solution of problem (1)–(3), changes little on each finite time interval under small changes of the boundary. In the case of problem (1), (2), (4), the vector U changes little. Since the solution depends continuously on the initial data ⁽⁵⁾, the problem is well posed in the classical sense. However, as S. L. Sobolev observed, the existence of solutions (10) in the case $\partial \mu_{k,l}^N / \partial z \neq 0$ shows that the problem is not suitable for describing the physical process for large t ⁽²⁾. Indeed, in this case U is expressed through $\text{grad } p$, and for sufficiently large t the difference $\|U(x_0, z_0, t) - U(x, z, t)\|$ is not an infinitely small quantity as (x, z, t) tends to (x_0, z_0, ∞) .

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