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S. Ya. Havinson

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Abstract

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MATHEMATICS

S. Ya. KHAVINSON

ON THE RUDIN-CARLESON THEOREM

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Rudin ⁽¹⁾ and Carleson ⁽²⁾ proved the following remarkable theorem (see also ⁽³⁾). If F is a closed subset of points of the circle $\Gamma : |\xi| = 1$, having measure zero, and $f(\xi)$ is an arbitrary continuous function on F , then there exists a function $\varphi(z)$, analytic in the disk $|z| < 1$, continuous in the closed disk $|z| \leq 1$, and coinciding on F with $f(\xi)$. Moreover, one may assume that $\max |\varphi(z)|$, for $|z| \leq 1$, coincides with $\max |f(\xi)|$, $\xi \in F$. An extension of this theorem to the case of functional algebras more general than the algebra of continuous analytic functions was given by Bishop ⁽⁴⁾. A profound strengthening of this theorem was obtained by Pelczyński ^(5,6), who also considered the space of continuous functions. We also point to a paper of Gamelin ⁽⁷⁾. In the present note we consider an analogue of the Rudin-Carleson theorem in the case of subspaces of an arbitrary Banach space X .

Theorem 1. *Let X be a Banach space, X^* its conjugate space, and suppose that the following conditions are fulfilled:*

1. $X^* = X_1^* \oplus X_2^*$, and the subspace X_2^* is weakly closed.
2. If $x_1^* \in X_1^*$ and $x_2^* \in X_2^*$, then $\|x_1^* + x_2^*\| \geq \|x_2^*\|$.
3. A closed subspace $Z \subset X$ is such that, if $x^*(z) = 0$, $z \in Z$, then $x^* \in X_1^*$.

Denote by X_2 the annihilator of X_2^* in the space X : $X_2 = (X_2^*)_\perp$, and let $Z_2 = Z \cap X_2$. Then $X/X_2 \simeq Z/Z_2$ ($\simeq$ is an isomorphic isometry).

Remark. Let us first of all explain why this theorem is a generalization of the Rudin-Carleson result. Let $X = C(\Gamma)$, the space of continuous complex-valued functions on Γ ; let Z be the subspace of $C(\Gamma)$ consisting of functions admitting analytic continuation into the disk $|z| < 1$. The space X^* consists, as is known, of all finite Borel measures on Γ . Let X_2^* consist of all those measures whose support is contained in F , and X_1^* of the measures concentrated on $\Gamma \setminus F$. Condition 1 is here, obviously, fulfilled. Condition 2 is fulfilled in exactly the same way, since in our case

$$\|x_1^* + x_2^*\| = \|x_1^*\| + \|x_2^*\|.$$

If some functional x^* is orthogonal to Z , then the measure defining it, by the F. and M. Riesz theorem, is absolutely continuous, and therefore its restriction to F is equal to zero, which means that condition 3 is fulfilled.

The subspace X_2 is composed of all functions equal to zero on F , and, as is easily seen, $X/X_2 \simeq C(F)$. The main assertion of the Rudin–Carleson theorem consists precisely in the fact that $Z/Z_2 \simeq C(F)$, and therefore follows from the theorem we have stated.

Proof of the theorem. For brevity denote $R = X/X_2$ and $R_1 = Z/Z_2$.

Define a mapping of R_1 into R in the following way. To the class $\alpha = \{z_0 + z_2\} \in R_1$, where z_0 is a fixed element of Z , and z_2 runs through Z_2 , we put in correspondence the class $\tilde{\alpha} = \{z_0 + x_2\}$, where x_2 runs through X_2 . The image of R_1 under our mapping will be denoted by $\tilde{R}_1$. The mapping of R_1 onto $\tilde{R}_1$ is one-to-one. Indeed, if $\alpha_1 \neq \alpha_2$, where $\alpha_1 = \{z'_0 + z_2\}$ and $\alpha_2 = \{z''_0 + z_2\}$, then $z'_0 - z''_0 \notin Z_2$, and therefore $z'_0 - z''_0 \notin X_2$, and consequently—

indeed, $\tilde{\alpha}_1 \neq \tilde{\alpha}_2$. It is now clear that the mapping R_1 onto $\tilde{R}_1$ is an isomorphism. Our next step will be to prove that $\tilde{R}_1$ is dense in R . The annihilator $X_2^\perp$ of the subspace X_2 in the space X^* coincides with X_2^* by the weak closedness of $X_2^* : X_2^\perp = [(X_2^*)^\perp]^\perp = X_2^*$ (see, for example, ⁽⁸⁾, Ch. II, § 9). By the Hahn–Banach theorem this allows us to identify R^* with X_2^* . Indeed, any functional $\varphi \in R^*$ may be regarded as defined on X and orthogonal to X_2 ; conversely, any $x_2^* \in X_2^*$ may be regarded as a functional on R , and it is also easy to see that if $\varphi \in R^*$ and $x_2^* \in X_2^*$ correspond to one another, then $\|\varphi\|_R = \|x_2^*\|_X$. This follows from the fact that, on the one hand, $|x_2^*(x)| = |\varphi(\beta)| = |\varphi(\{x + x_2\})| \leq \|\varphi\|_R \cdot \|\beta\| \leq \|\varphi\|_R \cdot \|x\|$, and hence $\|x_2^*\|_X \leq \|\varphi\|_R$; and, on the other hand,

$$\|\varphi\|_R = \sup_{\|\beta\| \leq 1} |\varphi(\beta)| = \sup_{\|x+x_2\| \leq 1} |x_2^*(x)| = \sup_{\|x+x_2\| \leq 1} |x_2^*(x+x_2)| \leq \sup_{\|t\| \leq 1} |x_2^*(t)| = \|x_2^*\|_X.$$

Thus, $R^* \simeq X_2^*$.

If now $\varphi \in R^*$ and $\varphi(\tilde{\alpha}) = 0$ for $\tilde{\alpha} \in \tilde{R}_1$, then this means that the corresponding functional $x_2^* \in X_2^*$ vanishes on Z , and therefore, by assumption 3, $x_2^* \equiv 0$. Hence $\varphi \equiv 0$, and $\tilde{R}_1$ is dense in R .

We shall now prove that $R_1 \simeq \tilde{R}_1$. Since $R_1 = Z/Z_2$ is a Banach space, it will follow from the last isometry that $\tilde{R}_1$ is complete; and this, together with the density of $\tilde{R}_1$ in R , leads to $R_1 = R \simeq \tilde{R}_1$. Consider the space Z^* . By the Hahn–Banach theorem one may suppose that Z^* consists of the same functionals $x^* \in X^*$, but now

$$\|x^*\|_Z = \inf_{q^* \in Z^\perp} \|x^* - q^*\| \leq \|x^*\|.$$

Here it is important to remember that condition 3 means the inclusion $Z^\perp \subset X_1^*$. Condition 2 now gives, for every functional x_2^* ,

$$\|x_2^*\|_Z = \inf_{q^* \in Z^\perp} \|x_2^* - q^*\| \geq \|x_2^*\|,$$

and consequently $\|x_2^*\|_Z = \|x_2^*\|$. Thus, X_2^* may be regarded as isometrically embedded in Z^* . Denote the image of X_2^* under this embedding by $\widetilde{X}_2^*$. We shall show that $\widetilde{X}_2^*$ is weakly closed in Z^* . By the Krein–Šmulian theorem (see, for example, ⁽⁹⁾, p. 465) it suffices to show that the intersection of $\widetilde{X}_2^*$ with an arbitrary closed ball S centered at the origin of Z^* is weakly closed. Let $y^* \in Z^*$ be a limit point of the set $\widetilde{X}_2^* \cap S$. This means that some generalized sequence $\{\tilde{x}_2^{*l}\}$ (l ranges over the set of indices defining the generalized sequence) of elements of $\widetilde{X}_2^* \cap S$ has y^* as its limit in the weak topology of the space Z^* . Let T be the closed ball in X^* of the same radius as S . The set $X_2^* \cap T$ is bicomact in the weak topology of X^* , and therefore the generalized sequence $\{x_2^{*l}\}$, contained in $X_2^* \cap T$ and corresponding to the sequence $\{\tilde{x}_2^{*l}\}$, has in $X_2^* \cap T$ a generalized limit point x_2^* . From the definition of the weak topologies in Z^* and X^* it follows that on Z the functionals y^* and x_2^* coincide, and the latter means that $y^* \in \widetilde{X}_2^*$. The closedness of $\widetilde{X}_2^*$ in the weak topology of Z^* is proved. Evidently, we have $(\widetilde{X}_2^*)^\perp = Z_2$ and $Z_2^\perp = \widetilde{X}_2^*$.

Now, for the coset $\alpha = \{z_0 + z_2\} \in R_1$ we obtain (again with the help of the Hahn–Banach theorem)

$$\|\alpha\| = \inf_{z_2 \in Z_2} \|z_0 + z_2\| = \sup_{\substack{x_2^* \in \widetilde{X}_2^* \\ \|x_2^*\| \leq 1}} |\tilde{x}_2^*(z_0)| = \sup_{\substack{x_2^* \in X_2^* \\ \|x_2^*\| \leq 1}} |x_2^*(z_0)|.$$

For the coset $\tilde{\alpha} = \{z_0 + x_2\} \in \widetilde{R}_1$, corresponding to α under the isomorphism of R_1 onto $\widetilde{R}_1$, we have

$$\|\tilde{\alpha}\| = \inf_{x_2 \in X_2} \|z_0 + x_2\| = \sup_{\substack{x_2^* \in X_2^* \\ \|x_2^*\| \leq 1}} |x_2^*(z_0)| = \|\alpha\|.$$

The isometry $R_1 \simeq \widetilde{R}_1$ is proved, and this completes the entire proof of the theorem.

Returning to the Rudin–Carleson theorem, let us note that from our theorem it follows only that the analytic function $\varphi(z)$, extending $f(\zeta)$, can be chosen so that

$$\max_{|z| \leq 1} |\varphi(z)| \leq \max_{\zeta \in F} |f(\zeta)| + \varepsilon,$$

where $\varepsilon > 0$ is arbitrary. The possibility of extending $f(\zeta)$ without increasing the norm does not follow directly from our theorem. In our terms, the possibility of

such an extension would mean that for every $z \in Z$ there is an element $z_2^0 \in Z_2$ such that

$$\|z + z_2^0\| = \inf \|z + z_2\|, \quad z_2 \in Z_2.$$

If D is a closed subset of $\Gamma \setminus F$, then the function $f(\zeta)$ on F can be extended to Γ continuously in such a way that the extending function is equal to zero on D . If, however, one restricts oneself only to functions $\varphi(z)$ analytic for $|z| < 1$, then for every $\varepsilon > 0$ there is an extension $\varphi(z)$ such that $|\varphi(z)| < \varepsilon$, $z \in D$. These facts carry over to the abstract situation considered in the following way.

Theorem 2. *Suppose that, under the hypotheses of Theorem 1, the subspace $Q \subset X_1^*$ is closed in the weak topology of X^* and $P = Q^\perp$. Then the image of P under the natural homomorphism $X \rightarrow X/X_2$ coincides with all of X/X_2 . In other words, for every $x \in X$ there exist $x_2 \in X_2$ and $p \in P$ such that $x = x_2 + p$. Moreover, for every $z \in Z$ and $\varepsilon > 0$ there exists $z_2 \in Z_2$ such that*

$$\sup_{q \in Q, \|q\| \leq 1} |q(z + z_2)| = \inf_{p \in P} \|z + z_2 + p\| < \varepsilon.$$

Proof. Let U be the operator mapping P into $X/X_2 = R$ under the natural homomorphism $X \rightarrow X/X_2$. To prove the first assertion of the theorem it is necessary, for the adjoint operator U^* , to establish the inequality

$$\|U^*(x_2^*)\| \geq m \|x_2^*\|$$

for some $m > 0$, since the space $R^* = X_2^*$. But

$$U^*(x_2^*) = x_2^* + q$$

with some $q \in P^\perp = Q$. Since $Q \subset X_1^*$, by assumption 2 of Theorem 1,

$$\|x_2^* + q\| \geq \|x_2^*\|.$$

To prove the second assertion of the theorem, we shall show that

$$\inf_{z_2 \in Z_2, p \in P} \|z + z_2 + p\| = \left\| \sup_{\|x^*\| \leq 1, x^* \in Z_2^\perp \cap Q} |x^*(z)| \right\| = 0.$$

Every functional $x^* \in Z_2^\perp$ has the form $x^* = x_2^* + z^*$, where $x_2^* \in X_2^*$, and $z^* \in Z^\perp \subset X_1^*$. Since $x^* \in Q$, we have $x_2^* + z^* = q \in Q \subset X_1^*$. Hence it follows that $x_2^* = 0$, and this completes the proof.

Moscow Civil Engineering Institute
named after V. V. Kuibyshev

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