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Abstract

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MATHEMATICS

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EMBEDDING THEOREMS FOR SOME CLASSES OF ABSTRACT FUNCTIONS

(Presented by Academician S. L. Sobolev on 23 I 1964)

We denote each point $\mathbf{x}$ of the n -dimensional Euclidean space R^n in the form of a pair $(\mathbf{x}_{n-s}, \mathbf{x}_s)$, where $\mathbf{x}_{n-s}(x_1, x_2, \dots, x_{n-s})$, $\mathbf{x}_s(x_{n-s+1}, x_{n-s+2}, \dots, x_n)$. In addition, let R^{n-s} be the $(n-s)$ -dimensional space of vectors $\mathbf{x}_{n-s}$; R^s be the s -dimensional space of vectors $\mathbf{x}_s$; $\bar{R}_0^n$ be the half-space $x_n \geq 0$; $x(x_1, x_2, \dots, x_{n-1}) \in R^{n-1}$; $\bar{R}_0^s$ be the half-space of points $x_n \geq 0$; $(x_{n-s+1}, x_{n-s+2}, \dots, x_{n-1}) \in R^{s-1}$.

We shall regard each function $f(\mathbf{x})$, defined in R^n , as a function of the vector variables $\mathbf{x}_{n-s}$ and $\mathbf{x}_s$. Under such consideration, to almost every vector $\mathbf{x}_{n-s}^{(0)}$ there corresponds an element of a certain abstract space—the function $f(\mathbf{x}_{n-s}^{(0)}, \mathbf{x}_s)$ of the variable vector $\mathbf{x}_s$.

From the collection of abstract functions defined in this way one constructs the abstract classes:

$$L_{\alpha; (p_1, p_2)}^l(\bar{R}_0^n), \quad W_{\alpha; (p_1, p_2)}^l(\bar{R}_0^n), \quad W_{(p_1, p_2)}^r(R^n), \quad B_{(p_1, p_2)}^r(R^n)$$

—analogues of the known classes $L_{\alpha; p}^l, W_{\alpha; p}^l, W_p^r, B_p^r$ ($1 \leq l \leq 7$).

I. Definition of the classes $L_{\alpha; (p_1, p_2)}^l(\bar{R}_0^n)$ (l an integer). We shall say that a function $f(\mathbf{x})$ belongs to the class $L_{\alpha; (p_1, p_2)}^l(\bar{R}_0^n)$ if:

- a) $f(\mathbf{x})$ has in $\bar{R}_0^n$ all generalized derivatives of order l ;
- b)

$$\|f\|_{L_{\alpha; (p_1, p_2)}^l(\bar{R}_0^n)} = \left\{ \int_{R^{n-s}} \left[\int_{\bar{R}_0^s} x_n^\alpha \sum_{|\bar{\alpha}|=l} |D^{\bar{\alpha}} f|^{p_1} d\bar{R}_0^s \right]^{p_2/p_1} dR^{n-s} \right\}^{1/p_2} < \infty.$$

II. Definition of the classes $W_{\alpha; (p_1, p_2)}^l(\bar{R}_0^n)$. We shall say that a function $f(\mathbf{x})$ belongs to $W_{\alpha; (p_1, p_2)}^l(\bar{R}_0^n)$ if:

a) $f(\mathbf{x}) \in L_{\alpha; (p_1, p_2)}^l(\bar{R}_0^n);$

b) $f(\mathbf{x}) \in L_{(p_1, p_2)}(\bar{R}_0^n).$

The norm in $W_{\alpha; (p_1, p_2)}^l(\bar{R}_0^n)$ is defined by the equality

$$\|f\|_{W_{\alpha; (p_1, p_2)}^l(\bar{R}_0^n)} = \|f\|_{L_{(p_1, p_2)}(\bar{R}_0^n)} + \|f\|_{L_{\alpha; (p_1, p_2)}^l(\bar{R}_0^n)}.$$

III. Definition of the classes $W_{(p_1, p_2)}^r(R^n)$.

1) r an integer. Put $\mathbf{r}(r, r, \dots, r)$. We shall say that a function

$$f(\mathbf{x}) \in W_{(p_1, p_2)}^r(R^n)$$

if:

a) $f(\mathbf{x}) \in L_{(p_1, p_2)}(R^n);$

b) $f(\mathbf{x}) \in L_{0; (p_1, p_2)}^r(R^n).$

The norm in $W_{(p_1, p_2)}^r(R^n)$ is defined by the equality

$$\|f\|_{W_{(p_1, p_2)}^r(R^n)} = \|f\|_{L_{(p_1, p_2)}(R^n)} + \|f\|_{L_{0; (p_1, p_2)}^r(R^n)}.$$

2) r nonintegral. Let $r = \bar{r} + \alpha$, where $0 < \alpha < 1$; $1 < p_1 < \infty$. Put

$$\mathbf{r}(\underbrace{r, \dots, r}_{n-s}, \underbrace{\bar{r}, \dots, \bar{r}}_s).$$

We shall say that a function $f(\mathbf{x}) \in W_{(p_1, p_2)}^r(R^n)$,

if

a) $f(x) \in W_{(p_1, p_2)}^{\bar{r}}(R^n)$, where $\bar{r}(\underbrace{\bar{r}, \bar{r}, \dots, \bar{r}}_n)$;

b)

$$\|\bar{D}^{\bar{\beta}} f\|_{\bar{W}_{(p_1, p_2)}^{-\alpha}(R^n)} = \sum_{j=n-s+1}^n \left\{ \int_{R^{n-s}} \left[\int_{R^s} \left(\int_0^\infty \frac{|\Delta_{j,h} D^{\bar{\beta}} f|^{p_1}}{h^{1+p_1\alpha}} dh \right) dR^s \right]^{p_2/p_1} dR^{n-s} \right\}^{1/p_2} < \infty,$$

where

$$\Delta_{j,h} D^{\bar{\beta}} f(x_1, \dots, x_j + h, \dots, x_n) - D^{\bar{\beta}} f(x_1, \dots, x_j, \dots, x_n); \quad |\vec{\beta}| = \bar{r}.$$

We define the norm in $W_{(p_1, p_2)}^r(R^n)$ by the equality

$$\|f\|_{W_{(p_1, p_2)}^r(R^n)} = \|f\|_{W_{(p_1, p_2)}^{\bar{r}}(R^n)} + \sum_{|\vec{\beta}|=\bar{r}} \|D^{\bar{\beta}} f\|_{\bar{W}_{(p_1, p_2)}^{-\alpha}(R^n)}.$$

IV. Definition of the classes $B_{(p_1, p_2)}^r(R^n)$.

1) r an integer. We shall say that $f(x) \in B_{(p_1, p_2)}^r(R^n)$ if:

a) $f(x) \in W_{(p_1, p_2)}^{\bar{\gamma}}(R^n)$, where $\bar{\gamma}(\underbrace{r-1, r-1, \dots, r-1}_n)$;

b)

$$\|D^{\bar{\beta}} f\|_{\bar{B}_{(p_1, p_2)}^1(R^n)} = \sum_{j=n-s+1}^n \left\{ \int_{R^{n-s}} \left[\int_{R^s} \left(\int_0^\infty \frac{|\Delta_{j,h}^2 D^{\bar{\beta}} f|^{p_1}}{h^{1+p_1}} dh \right) dR^s \right]^{p_2/p_1} dR^{n-s} \right\}^{1/p_2} < \infty,$$

where $|\bar{\beta}| = r-1$;

$$\Delta_{j,h}^2 D^{\bar{\beta}} f = D^{\bar{\beta}} f(x_1, \dots, x_j-h, \dots, x_n) - 2D^{\bar{\beta}} f(x_1, \dots, x_j, \dots, x_n) + D^{\bar{\beta}} f(x_1, \dots, x_j+h, \dots, x_n).$$

We define the norm in $B_{(p_1, p_2)}^r(R^n)$ by the equality

$$\|f\|_{B_{(p_1, p_2)}^r(R^n)} = \|f\|_{W_{(p_1, p_2)}^{\bar{\gamma}}(R^n)} + \sum_{|\bar{\beta}|=r-1} \|D^{\bar{\beta}} f\|_{\bar{B}_{(p_1, p_2)}^1(R^n)}.$$

2) r noninteger. Put

$$r(\underbrace{\bar{r}, \bar{r}, \dots, \bar{r}}_{n-s}, \underbrace{r, r, \dots, r}_s).$$

In this case we shall assume that $f(x) \in B_{(p_1, p_2)}^r(R^n)$ if $f(x) \in W_{(p_1, p_2)}^r(R^n)$.

Theorem 1. If $\alpha - kp_1 + 1 > 0$, then

$$W_{\alpha; (p_1, p_2)}^l(\bar{R}_0^n) \rightarrow W_{\alpha - kp_1; (p_1, p_2)}^{l-k}(\bar{R}_0^n).$$

Theorem 2. If $\alpha - p_1 l + 1 < 0$, then

$$W_{\alpha; (p_1, p_2)}^l(\bar{R}_0^n) \rightarrow B_{(p_1, p_2)}^{\bar{\gamma}}(R^{n-1}),$$

where

$$\bar{\gamma} \left(\underbrace{\left[l - \frac{\alpha+1}{p_1} \right], \dots, \left[l - \frac{\alpha+1}{p_1} \right]}_{n-s}, \underbrace{l - \frac{\alpha+1}{p_1}, \dots, l - \frac{\alpha+1}{p_1}}_{s-1} \right),$$

$[\alpha]$ is the integer part of the number α .

Theorem 3. Let $1 \leq m \leq n-1$, $1 < p < p_1 < p_2 < \infty$,

$$r = l - \frac{n-1}{p} + \frac{m}{p_1} + \frac{n-m-1}{p_2};$$

then

$$W_p^l(R^{n-1}) \rightarrow W_{(p_1, p_2)}^r(R^{n-1}),$$

where $r(\underbrace{\bar{r}, \dots, \bar{r}}_{n-s}, \underbrace{r, \dots, r}_{s-1})$, l is nonintegral ($W_{\alpha_1}^{\beta_1} \rightarrow W_{\alpha_2}^{\beta_2}$ denotes the embedding of $W_{\alpha_1}^{\beta_1}$ in $W_{\alpha_2}^{\beta_2}$ (see the survey article of S. M. Nikol'skii⁽⁸⁾)).

Theorems 1 and 2, as is not difficult to see, can be obtained from the corresponding theorems of S. V. Uspenskii⁽⁴⁾.

We shall outline the proof of Theorem 3. Consider the function

$$F = kx_n \int_{R^{n-1}} \frac{v(t, x_n) dt}{[(t-x)^2 + x_n^2]^{n/2}}.$$

From the preceding theorems it follows that, in order to prove Theorem 3, it is sufficient to establish the validity of the inequality

$$\|F\|_{L_{p_1(\nu-r)-1; (p_1, p_2)}^\nu(\bar{R}_0^n)} \leq c \|f\|_{W_p^l(R^{n-1})}. \quad (1)$$

Using the estimate

$$|F^\nu(x, x_n)| \leq c \sum_{|\bar{\alpha}| \leq \nu} \int_{R^{n-1}} \frac{|v^\nu(t, x_n)| x_n}{[(t-x)^2 + x_n^2]^{n/2}} dt + c \sum_{|\bar{\alpha}| \leq \nu-1} \int_{R^{n-1}} \frac{|v^{\nu-1}(t, x_n)| dt}{[(t-x)^2 + x_n^2]^{n/2}} = c \left(\sum_{|\bar{\alpha}| \leq \alpha} I_{1, \bar{\alpha}} + \sum_{|\bar{\alpha}| \leq \alpha-1} I_{2, \bar{\alpha}} \right)$$

and Minkowski's inequality, we obtain

$$\|F\|_{L_{p_1(\nu-r)-1; (p_1, p_2)}^\nu(\bar{R}_0^n)} \leq c \left(\sum_{|\bar{\alpha}| \leq \nu} \|I_{1, \bar{\alpha}}\|_{L_{p_1(\nu-r)-1; (p_1, p_2)}^\nu(\bar{R}_0^n)} + \sum_{|\bar{\alpha}| \leq \nu-1} \|I_{2, \bar{\alpha}}\|_{L_{p_1(\nu-r)-1; (p_1, p_2)}^\nu(\bar{R}_0^n)} \right). \quad (2)$$

Let us estimate one of the terms, for example

$$\|I_{1, \bar{\alpha}}\|_{L_{p_1(\nu-r)-1; (p_1, p_2)}^\nu(\bar{R}_0^n)} = c \left\{ \int_{R^{n-m-1}} \left[\int_{\bar{R}_0^{m+1}} x_n^{p_1(\nu-r)-1} \left(\int_{R^{n-1}} \frac{|v^\nu(t, x_n)| x_n dt}{[(t-x)^2 + x_n^2]^{n/2}} \right)^{p_1} d\bar{R}_0^{m+1} \right]^{p_2/p_1} dR^{n-m-1} \right\}$$

Applying Hölder's inequality and simplifying, we shall have

$$\int_{\bar{R}_0^{m+1}} x_n^{p_1(\nu-r)-1} \left(\int_{R^{n-1}} \frac{|v^\nu(t, x_n)| x_n dt}{[(t-x)^2 + x_n^2]^{n/2}} d\bar{R}_0^{m+1} \right)^{p_1} \leq c \|f\|_{W_p^l}^{p_1-p} \int_{\bar{R}_0^{m+1}} x_n^k \left(\int_{R^{n-1}} \frac{|v^\nu|^p dt}{[(t-x)^2 + x_n^2]^{n/2}} \right) d\bar{R}_0^{m+1},$$

where $k = p_1(l - r) + p(\nu - l) - (n - 1)(p_1/p - 1)$.

Taking (3) into account and making the necessary transformations, we obtain

$$\|I_{1,\alpha}\|_{L_{p_1(\nu-r)-1;(p_1,p_2)}^\nu(\bar{R}_0^n)} \leq c \|f\|_{W_p^l}^{1-p/p_1} \left\{ \int_{R^{n-m-1}} \left[\int_{\bar{R}^{m+1}} x_n^{k-(1-p_1/p_2)(n-m-1)} \left(\int_{R^{n-1}} \frac{|v^\nu|^p dt}{[(t-x)^2 + x_n^2]^{n/2}} \right) d\bar{R}_0^{m+1} \right] dR^{n-m-1} \right\}$$

The estimates hold:

$$\int_{\bar{R}_0^n} x_n^{p(\nu-l)-1} |v^\nu(t, x_n)|^p d\bar{R}_0^n \leq c \|f\|_{W_p^l(R^{n-1})}^p,$$

$$\int_{R^{n-m-1}} \left[\int_{\bar{R}_0^{m+1}} x_n^{k-(1-p_1/p_2)(n-m-1)} \left(\int_{R^{n-1}} \frac{|v^\nu|^p dt}{[(t-x)^2 + x_n^2]^{n/2}} \right) d\bar{R}_0^{m+1} \right] dR^{n-m-1} \leq \int_{\bar{R}_0^n} x_n^{k-(1-p_1/p_2)(n-m-1)} d\bar{R}_0^n$$

therefore

$$\|I_{1,\alpha}^\rightarrow\|_{L_{p_1(\nu-r)-1;(p_1,p_2)}^\nu(\bar{R}_0^n)} \leq c \|f\|_{W_p^l(R^{n-1})}. \quad (4)$$

Similarly one can show that

$$\|I_{2,\alpha}^\rightarrow\|_{L_{(p_1(\nu-r)-1;(p_1,p_2))}^\nu(\bar{R}_0^n)} \leq c \|f\|_{W_p^l(R^{n-1})}. \quad (5)$$

From (2), (4), and (5) we obtain (1).

Hence it is not difficult to obtain the required result. Theorem 3 establishes the validity of the hypothesis of S. L. Sobolev and S. M. Nikol'skii for the generalized fractional spaces W_p^l of S. L. Sobolev. Works (⁹, ¹¹, ¹²) are also devoted to this hypothesis.

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Note: Figure translations are in progress. See original paper for figures.

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