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Abstract

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MATHEMATICS

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THE OBJECT OF A CENTRO-PROJECTIVE CONNECTION ON A MANIFOLD WITH AN ALMOST COMPLEX STRUCTURE

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1. Let $\mathfrak{M}_{2n}$ be a $2n$ -dimensional differentiable manifold with an almost complex structure, i.e., on $\mathfrak{M}_{2n}$ there is defined a tensor F_j^i ($i, j, k, l = 1, 2, \dots, 2n$) such that $F_j^i F_i^k = -\delta_j^k$. Suppose that on $\mathfrak{M}_{2n}$ an object of centro-projective connection of a differential neighborhood of the third order $(\Gamma_{jk}^i, \Gamma_{jk})$ and a copuncture a_i are given ⁽¹⁾.

The object with components

$$\begin{aligned}\tilde{\Gamma}_{jk}^i &= \Gamma_{jk}^i + F_j^s F_{s,k}^i, \\ \tilde{\Gamma}_{jk} &= \Gamma_{jk} - a_i F_j^s F_{s,k}^i + (\delta_j^s - F_j^s) D_k a_s,\end{aligned}\tag{1}$$

where $D_k a_s = \partial a_s / \partial x^k - a_p \Gamma_{sk}^p - \Gamma_{sk}$, and $F_{s,k}^i$ denotes covariant differentiation in the connection Γ_{jk}^i , defines on $\mathfrak{M}_{2n}$ a centro-projective connection, the transport of a puncture in which along a curve L from a point M to a point M_1 is carried out as follows: a puncture u^i at the point M is transformed into the puncture

$$v^i = \frac{F_k^i u^k}{-a_p F_j^p u^j + a_p u^p + 1},\tag{2}$$

then v^i is transported along the curve L from the point M to the point M_1 in the centro-projective connection $(\Gamma_{jk}^i, \Gamma_{jk})$ and at the point M_1 is transformed into the puncture

$$\tilde{u}^i = \frac{-F_j^i \tilde{v}^j}{a_p F_j^p \tilde{v}^j + a_p \tilde{v}^p + 1}$$

by the transformation inverse to (2).

The object $(\tilde{\Gamma}_{jk}^i, \tilde{\Gamma}_{jk})$ is called the centro-projective connection conjugate to the connection $(\Gamma_{jk}^i, \Gamma_{jk})$ with respect to the copuncture a_i . From (1) it is seen that the subobject $\tilde{\Gamma}_{jk}^i$ of the object of the conjugate centro-projective connection $(\tilde{\Gamma}_{jk}^i, \tilde{\Gamma}_{jk})$ is an affine connection conjugate to the affine connection Γ_{jk}^i ⁽²⁾, which forms the subobject of the original object of centro-projective connection $(\Gamma_{jk}^i, \Gamma_{jk})$.

Theorem 1. *If the centro-projective connection $(\Gamma_{jk}^i, \Gamma_{jk})$ is attached to an affine connection Γ_{jk}^i and a_i is a corresponding copuncture covariantly constant in this connection, then the conjugate centro-projective connection $(\tilde{\Gamma}_{jk}^i, \tilde{\Gamma}_{jk})$ is also attached to the connection $\tilde{\Gamma}_{jk}^i$ conjugate to the affine connection Γ_{jk}^i .*

Proof. Since $D_j a_k = \partial a_k / \partial x^j - a_p \Gamma_{kj}^p - \Gamma_{kj} = 0$, we have

$$\tilde{\Gamma}_{jk} = \Gamma_{jk} - a_i F_j^s F_{s,k}^i = \Gamma_{jk} - a_i (\tilde{\Gamma}_{jk}^i - \Gamma_{jk}^i) = \frac{\partial a_j}{\partial x^k} - a_p \tilde{\Gamma}_{jk}^p.$$

The connection $(\tilde{\Gamma}_{jk}^i, \tilde{\Gamma}_{jk})$ is attached to the affine connection $\tilde{\Gamma}_{jk}^i$, and the copuncture a_k is covariantly constant in this connection.

Let us note that, for a covariant tensor of degree one, the analogous property does not hold: a tensor, being covariantly constant in an affine connection—

of Γ_{jk}^i , is not at all obliged to be covariantly constant in the conjugate connection $\tilde{\Gamma}_{jk}^i$.

2. The object of a centro-projective connection

$$\left\{ \Gamma_{jk}^i, \quad \Gamma_{jk} = -\frac{1}{2n+1} \left(\frac{\partial \Gamma_{mj}^m}{\partial x^k} - \Gamma_{ml}^m \Gamma_{jk}^l \right) \right\},$$

defined by the copuncture $-\frac{1}{2n+1} \Gamma_{ml}^m$, will be called left-invariantly associated with the affine connection Γ_{jk}^i , and the object of a centro-projective connection defined by the copuncture $-\frac{1}{2n+1} \Gamma_{lm}^m$, right-invariantly associated with the affine connection Γ_{jk}^i . If the left- and right-invariantly associated objects of a centro-projective connection coincide, then we shall say that the object of a centro-projective connection $(\Gamma_{jk}^i, \Gamma_{jk})$ is invariantly associated with Γ_{jk}^i . It is clear that both objects of a centro-projective connection associated with a symmetric affine connection Γ_{jk}^i coincide.

The connection $\tilde{\Gamma}_{jk}^i$ conjugate to the symmetric affine connection Γ_{jk}^i is, generally speaking, nonsymmetric, and therefore the objects of a centro-projective connection left- and right-invariantly associated with it need not coincide.

Theorem 2. If the connection $(\Gamma_{jk}^i, \Gamma_{jk})$, invariantly associated with a symmetric affine connection Γ_{jk}^i , then the conjugate centro-projective connection $(\tilde{\Gamma}_{jk}^i, \tilde{\Gamma}_{jk})$ is left-invariantly associated with $\tilde{\Gamma}_{jk}^i$.

Proof. From Theorem 1 it follows that the copunctor $-\frac{1}{2n+1}\Gamma_{kj}^k$ is covariantly constant in the connection $(\tilde{\Gamma}_{jk}^i, \tilde{\Gamma}_{jk})$, and since $\tilde{\Gamma}_{kj}^k = \Gamma_{kj}^k$, everything is proved.

Consider the tensor $B_j = F_s^l F_{j,l}^s$. A symmetric affine connection Γ_{jk}^i for which $B_j = 0$ is called a normal affine connection. The tensor B_j itself we shall call the normal tensor of the connection Γ_{jk}^i .

Theorem 3. Let $(\Gamma_{jk}^i, \Gamma_{jk})$ be a centro-projective connection invariantly associated with a symmetric affine connection Γ_{jk}^i . In order that the conjugate centro-projective connection $(\tilde{\Gamma}_{jk}^i, \tilde{\Gamma}_{jk})$ be invariantly associated with $\tilde{\Gamma}_{jk}^i$, it is necessary and sufficient that the normal tensor B_j of the connection Γ_{jk}^i be covariantly constant in the conjugate connection $\tilde{\Gamma}_{jk}^i$.

3. The connection

$$\gamma_{jk}^i = \frac{1}{2}(\tilde{\Gamma}_{jk}^i + \tilde{\Gamma}_{kj}^i), \quad \gamma_{jk} = \frac{1}{2}(\tilde{\Gamma}_{jk} + \tilde{\Gamma}_{kj}) \quad (3)$$

is called the symmetrically conjugate centro-projective connection for the connection $(\Gamma_{jk}^i, \Gamma_{jk})$.

Theorem 4. Let $(\Gamma_{jk}^i, \Gamma_{jk})$ be associated with a symmetric affine connection Γ_{jk}^i , i.e. $\Gamma_{kj}^i = \Gamma_{jk}^i$, and suppose there exists a copunctor a_i such that $\Gamma_{jk} = \partial a_j / \partial x^k - a_p \Gamma_{jk}^p$. In order that the complete curvature object of the connection $(\Gamma_{jk}^i, \Gamma_{jk})$ be equal to zero, it is necessary and sufficient that the symmetrically conjugate centro-projective connection $(\gamma_{jk}^i, \gamma_{jk})$ be associated with the affine connection γ_{jk}^i by means of the copunctor a_i .

Proof. The necessity is obvious.

Sufficiency. Let $\gamma_{jk} = \partial a_j / \partial x^k - a_p \gamma_{jk}^p$. Taking into account relation (3), we obtain

$$\frac{1}{2}\tilde{\Gamma}_{jk} + \frac{1}{2}\tilde{\Gamma}_{kj} = \frac{\partial a_j}{\partial x^k} - \frac{1}{2}a_p \tilde{\Gamma}_{jk}^p - \frac{1}{2}a_p \tilde{\Gamma}_{kj}^p,$$

or $\partial a_k / \partial x^j = \partial a_j / \partial x^k$.

Theorem 5. Let a_j be a concircular and $(\Gamma_{jk}^i = \Gamma_{kj}^i, \Gamma_{jk}^i)$ a centro-projective connection in which this concircular is covariantly constant. If $(\gamma_{jk}^i, \gamma_{jk}^i)$ is the symmetrically conjugate connection for $(\Gamma_{jk}^i, \Gamma_{jk}^i)$, and $(\tilde{\gamma}_{jk}^i, \tilde{\gamma}_{jk}^i)$ is the conjugate centro-projective connection for the connection $(\gamma_{jk}^i, \gamma_{jk}^i)$, then the complete torsion object $(\tilde{S}_{jk}^i, \tilde{S}_{jk}^i)$ of the connection $(\tilde{\gamma}_{jk}^i, \tilde{\gamma}_{jk}^i)$ does not depend on the choice of the original connection $(\Gamma_{jk}^i, \Gamma_{jk}^i)$ and is equal to

$$\tilde{S}_{jk}^i = 2t_{jk}^i;$$

$$\tilde{S}_{jk} = 2a_i t_{kj}^i + \frac{1}{2} (\delta_j^s - F_j^s) \left(\frac{\partial a_s}{\partial x^k} - \frac{\partial a_k}{\partial x^s} \right) - \frac{1}{2} (\delta_k^s - F_k^s) \left(\frac{\partial a_s}{\partial x^j} - \frac{\partial a_j}{\partial x^s} \right),$$

where t_{jk}^i is the torsion tensor of the almost complex structure.

4. A normal centro-projective connection is defined as a symmetric centro-projective connection invariantly attached to a normal affine connection.

Theorem 6. If Γ_{jk}^i is an equiaffine connection and γ_{jk}^i is an affine connection symmetrically conjugate to it, then the centro-projective connection invariantly attached to γ_{jk}^i will be normal if and only if the normal tensor B_m of the equiaffine connection Γ_{jk}^i is a gradient.

Proof. Necessity. $\gamma_{mj}^m = \Gamma_{mj}^m - 1/2 B_j$ and $\partial \gamma_{mj}^m / \partial x^k = \partial \gamma_{mk}^m / \partial x^j$; moreover, $\partial \Gamma_{mj}^m / \partial x^k = \partial \Gamma_{mk}^m / \partial x^j$, and therefore $\partial B_j / \partial x^k = \partial B_k / \partial x^j$ and $B_j = \partial \varphi / \partial x^j$, where φ is an arbitrary scalar function.

Sufficiency. a) We shall prove that the normal tensor $\tilde{B}_l$ of the connection γ_{jk}^i is equal to zero.

$$\begin{aligned} \tilde{B}_l &= F_m^k \left(\frac{\partial F_l^m}{\partial x^k} + F_l^p \gamma_{pk}^m - F_p^m \gamma_{lk}^p \right) = \\ &= \frac{1}{2} F_m^k \left(\frac{\partial F_l^m}{\partial x^k} + F_l^p \tilde{\Gamma}_{pk}^m - F_p^m \tilde{\Gamma}_{lk}^p \right) + \frac{1}{2} F_m^k \left(\frac{\partial F_l^m}{\partial x^k} + F_l^p \tilde{\Gamma}_{kp}^m - F_p^m \tilde{\Gamma}_{kl}^p \right) = \\ &= \frac{1}{2} B_l + \frac{1}{2} F_m^k F_l^p F_p^s F_{s,k}^m - \frac{1}{2} F_m^k F_p^m F_l^s F_{s,k}^p + \\ &+ \frac{1}{2} B_l + \frac{1}{2} F_m^k F_l^p F_k^s F_{s,p}^m - \frac{1}{2} F_m^k F_p^m F_k^s F_{s,l}^p = \\ &= B_l - \frac{1}{2} B_l - \frac{1}{2} B_l - \frac{1}{2} F_l^p F_{s,p}^s + \frac{1}{2} F_k^s F_{s,l}^k = 0. \end{aligned}$$

- b) We shall prove that $\gamma_{jk} = \gamma_{kj}$:

$$\frac{\partial \gamma_{mk}^m}{\partial x^j} - \frac{\partial \gamma_{mj}^m}{\partial x^k} = \frac{\partial \Gamma_{mk}^m}{\partial x^j} - \frac{\partial \Gamma_{mj}^m}{\partial x^k} - \frac{1}{2} \left(\frac{\partial B_k}{\partial x^j} - \frac{\partial B_j}{\partial x^k} \right) = 0.$$

Theorem 7. If $(\Gamma_{jk}^i, \Gamma_{jk})$ is a normal centro-projective connection, then the centro-projective connection $(\gamma_{jk}^i, \gamma_{jk})$ symmetrically conjugate to it is also normal.

The proof follows from Theorem 4, taking into account that $\gamma_{mk}^m = \Gamma_{mk}^m$. The author expresses gratitude to V. G. Lemlein for valuable suggestions.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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