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Abstract

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MATHEMATICS

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STATIONARY RESONANT REGIMES OF CERTAIN ROTATIONAL MOTIONS

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§ 1. Statement of the problem. In papers ⁽¹⁻⁹⁾ the asymptotics of certain rotational motions with one degree of freedom, depending on slowly varying parameters, was considered. In ⁽⁴⁻⁶⁾ equations were obtained which describe, in the first and second approximations (with an error of order ε and ε^2 , respectively, where $\varepsilon > 0$ is a small parameter), the slow variation of the energy of the perturbed motion; the phase and period of the perturbed motion were also found, as well as the first approximations for the coordinate and velocity. However, in ⁽⁴⁻⁶⁾ the cases in which the perturbation depends explicitly on time were not considered. In the present paper, based on the methods developed in ⁽⁷⁾, the case is investigated in which the perturbation depends periodically on time, and the question of the existence and stability (in a certain generalized sense) of stationary resonant regimes of such systems is considered.

Let the rotational motion be described by a system of the form

$$\begin{aligned} \frac{d}{dt} [m(x)\dot{y}] + Q(x, y) &= \varepsilon f(x, y, \dot{y}, \vartheta), & \dot{x} &= \varepsilon X(x, y, \dot{y}, \vartheta), \\ \dot{\vartheta} &= \nu(x) + \varepsilon \Theta(x, y, \dot{y}, \vartheta). \end{aligned} \quad (1)$$

Here y is a one-dimensional coordinate; $m(x)$ is the mass; $x = (x_1, \dots, x_n)$ is a collection of slowly varying parameters; $Q(x, y)$ is the potential force causing the rotation; f is a small nonlinear perturbation; ϑ is the phase of the external perturbing force. All functions entering into (1) are assumed to be periodic in y and ϑ with period 2π , and Q has zero mean value in y .

§ 2. Main results. With the aid of a certain change of variables (see, for example, ⁽⁴⁻⁶⁾), system (1) can be rewritten in the form

$$\dot{E} = \varepsilon G(x, y, E, \vartheta), \quad \dot{x} = \varepsilon X \left(x, y, \sqrt{\frac{2}{m}(E - V)}, \vartheta \right), \quad (2)$$

$$\dot{\psi} = \omega(E, x) + \varepsilon \Psi(E, x, y, \vartheta), \quad \dot{\vartheta} = \nu(x) + \varepsilon \Theta \left(x, y, \sqrt{\frac{2}{m}(E - V)}, \vartheta \right),$$

where

$$\frac{\partial V}{\partial y} \equiv Q; \quad T = \int_0^{2\pi} \frac{dy}{\sqrt{\frac{2}{m}(E - V)}} \text{ is the period of rotation; } \quad \omega = \frac{2\pi}{T};$$

y is determined by integrals of the unperturbed system; E is the energy; ψ is the phase of rotation;

$$G = \frac{1}{m} \left(-E \frac{\partial m}{\partial x} + \frac{\partial m V}{\partial x} \right) X \left(x, y, \sqrt{\frac{2}{m}(E - V)}, \vartheta \right) + \sqrt{\frac{2}{m}(E - V)} f \left(x, y, \sqrt{\frac{2}{m}(E - V)}, \vartheta \right),$$

$$\Psi = \frac{2\pi}{T} \left[\int_{y_0}^y \frac{\frac{E}{m^2} \frac{\partial m}{\partial x} + \frac{\partial}{\partial x} \left(\frac{V}{m} \right)}{\left(\frac{2}{m}(E - V) \right)^{3/2}} d\eta - \frac{1}{T} \int_0^{2\pi} \frac{\frac{E}{m^2} \frac{\partial m}{\partial x} + \frac{\partial}{\partial x} \left(\frac{V}{m} \right)}{\left(\frac{2}{m}(E - V) \right)^{3/2}} dy \int_{y_0}^y \frac{d\eta}{\sqrt{\frac{2}{m}(E - V)}} \right] \times$$

$$\times X \left(x, y, \sqrt{\frac{2}{m}(E - V)}, \vartheta \right) +$$

$$+ \frac{2\pi}{mT} \left[\frac{1}{T} \int_0^{2\pi} \frac{dy}{\left(\frac{2}{m}(E - V) \right)^{3/2}} \int_{y_0}^y \frac{d\eta}{\sqrt{\frac{2}{m}(E - V)}} - \int_{y_0}^y \frac{d\eta}{\left(\frac{2}{m}(E - V) \right)^{3/2}} \right] G(E, x, y, \vartheta).$$

We shall say that a resonance characterized by relatively prime integers p and q occurs in the system if, for certain values of the parameters E, x , the equality $p\omega = q\nu$ is satisfied. Passing from the variables ψ, ϑ to the variables $\varphi = \vartheta - \frac{p}{q}\psi$, $\beta = \frac{1}{q}\psi$, we rewrite (2) in the form

$$\dot{E} = \varepsilon G(E, x, y, \varphi + p\beta), \quad \dot{x} = \varepsilon X \left(x, y, \sqrt{\frac{2}{m}(E - V)}, \varphi + p\beta \right),$$

$$\dot{\varphi} = \lambda(E, x) + \varepsilon \Phi(E, x, y, \varphi + p\beta), \quad \dot{\beta} = \Omega(E, x) + \varepsilon B(E, x, y, \varphi + p\beta), \quad (3)$$

where $\lambda = \nu - \frac{p}{q}\omega$, $\Omega = \frac{1}{q}\omega$, $\Phi = \Theta - \frac{p}{q}\Psi$, $B = \frac{1}{q}\Psi$.

We introduce, in the following way, the mean values of the functions appearing on the right-hand sides of (3), for fixed values of the parameters E_0, x_0 :

$$G_1(E_0, x_0, \varphi) = \quad (4)$$

$$= \frac{1}{qT(E_0, x_0)} \int_{y_0}^{y_0+2\pi q} G \left(E_0, x_0, y, \varphi + p\beta_0 + \nu_0 \int_{y_0}^y \frac{d\eta}{\sqrt{\frac{2}{m}(E_0 - V)}} \right) \frac{dy}{\sqrt{\frac{2}{m}(E_0 - V)}},$$

where $\beta_0 = \beta|_{t=t_0}$. The mean values of the functions X, Φ and of the derivatives $\partial G/\partial x_0, \partial G/\partial E_0$, etc., are defined analogously.

We apply to system (3) the scheme developed in (7) for systems of a more general form. The resonant values of the energy E_0 , of the parameters x_0 , and of the detuning φ_0 in the zeroth approximation in ε are found from the system

$$G_1(E_0, x_0, \varphi_0) = 0, \quad X_1(E_0, x_0, \varphi_0) = 0, \quad \lambda(E_0, x_0) = 0. \quad (5)$$

The corrections to the coordinates of the resonant point $\delta E, \delta x, \delta \varphi$ in the first approximation are found from the linear system of equations

$$\begin{aligned} \frac{\partial G_1}{\partial E_0} \delta E + \frac{\partial G_1}{\partial x_0} \delta x + \frac{\partial G_1}{\partial \varphi_0} \delta \varphi &= 0, & \frac{\partial X_1}{\partial E_0} \delta E + \frac{\partial X_1}{\partial x_0} \delta x + \frac{\partial X_1}{\partial \varphi_0} \delta \varphi &= 0, \\ \frac{\partial \lambda}{\partial E_0} \delta E + \frac{\partial \lambda}{\partial x_0} \delta x + \Phi_1(E_0, x_0, \varphi_0) &= 0. \end{aligned} \quad (6)$$

We introduce, in accordance with (7), the characteristic equation

$$\det(A - kE) = 0, \quad (7)$$

where E is the identity matrix of the corresponding dimension, and A is a matrix of the form

$$A = \begin{pmatrix} \varepsilon \frac{\partial G_1}{\partial E_0} & \varepsilon \frac{\partial G_1}{\partial x_0} & \varepsilon \frac{\partial G_1}{\partial \varphi_0} \\ \varepsilon \frac{\partial X_1}{\partial E_0} & \varepsilon \frac{\partial X_1}{\partial x_0} & \varepsilon \frac{\partial X_1}{\partial \varphi_0} \\ \frac{\partial \lambda}{\partial E_0} + \varepsilon \left(\frac{\partial \Phi_1}{\partial E_0} + \frac{\partial^2 \lambda}{\partial E_0 \partial x_0} \delta x + \frac{\partial^2 \lambda}{\partial E_0^2} \delta E \right) & \frac{\partial \lambda}{\partial x_0} + \varepsilon \left(\frac{\partial \Phi_1}{\partial x_0} + \frac{\partial^2 \lambda}{\partial x_0^2} \delta x + \frac{\partial^2 \lambda}{\partial x_0 \partial E_0} \delta E \right) & \varepsilon \frac{\partial \Phi_1}{\partial \varphi_0} \end{pmatrix}.$$

We require that systems (5) and (6) be solvable and that all roots (7) have negative real parts:

$$\operatorname{Re} k < 0. \quad (8)$$

We shall call condition (8) the stability condition for the stationary regime (3). Here stability is understood in the following sense: from results (7) it follows that for arbitrarily large $T > 0$ and arbitrarily small $\xi > 0$ there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ there exists a $\delta(\varepsilon)$ such that from the condition

$$\max |E(t_0) - E_0, x(t_0) - x_0, \varphi(t_0) - \varphi_0| < \delta$$

for all $t_0 \leq t \leq T$ there follows the inequality

$$\max |E(t) - E_0, x(t) - x_0, \varphi(t) - \varphi_0| < \xi.$$

The statement just given can be refined: if the initial data for (3) are given in some ε -neighborhood of the resonant point E_0, x_0, φ_0 , then on a time interval of order $\varepsilon^{-1/2}$ the solutions of (3) will not leave some neighborhood of the resonant point of size of order $\varepsilon^{1/2}$.

§ 3. The case of slow time. Let the only slowly varying parameter be the slow time $\tau = \varepsilon t$. Then the coordinates of stationary resonant points are found from the equations

$$\frac{\partial \lambda / \partial \tau}{\partial \lambda / \partial E_0} + G_1(E_0, \varphi_0, \tau) = 0, \quad \lambda(E_0, \tau) = 0,$$

where in this case $E_0 = E_0(\tau), \varphi_0 = \varphi_0(\tau)$. The conditions ensuring (8) take the form:

$$\left(\frac{\partial G_1}{\partial E_0} + \frac{\partial \Phi_1}{\partial \varphi_0} \right)_{\tau=\tau_0} < 0, \quad \left(\frac{\partial \lambda}{\partial E_0} \frac{\partial G_1}{\partial \varphi_0} \right)_{\tau=\tau_0} < 0.$$

The principal assertion made in § 2 is also valid in this case.

§ 4. The case of large energies. Let us consider the case of large energies, important for applications, using as an example a perturbation of the special form:

$$\ddot{y} + Q(y) = \varepsilon[f(\nu t) - \lambda \dot{y}]. \quad (9)$$

Here Q satisfies the conditions listed in § 1, f is periodic in νt with period 2π , and $\lambda > 0$. The asymptotics at large energies for (9) was studied by other methods by N. N. Moiseev in ⁽¹⁾, where stationary regimes were investigated under the assumption that the parameter values differ from the resonant values by quantities of order ε .

Applying the results of § 2 to (9), we obtain the equations for determining the coordinates of the stationary point in the zero approximation in the form

$$-\frac{p}{q}\sqrt{2E_0} + \nu + \frac{1}{\sqrt{2E_0}}\frac{p\bar{V}}{q} = 0, \quad \bar{V} = \frac{1}{2\pi} \int_0^{2\pi} V(y) dy,$$

$$-2\lambda E_0 + \sqrt{2E_0}f(\varphi_0) + 2\lambda\bar{V} + \pi\nu q f'(\varphi_0) + \frac{1}{\sqrt{2E_0}} \left(\frac{2\pi^2\nu^2 q^2 f''(\varphi_0)}{3} - \bar{V}f(\varphi_0) \right) = 0. \quad (10)$$

(Here and below it is assumed that $y|_{t=t_0} = O(\varepsilon)$, $\varphi = \vartheta - \frac{p}{q}\psi + p\beta_0$.)

The stability condition for the stationary resonant regime (10) has the form:

$$f'(\varphi_0) + \frac{\pi\nu q}{\sqrt{2E_0}} f''(\varphi_0) > 0.$$

§ 5. **Examples.** As a first example, consider the mathematical pendulum in a rotational regime, studied by another method in ⁽²⁾. The equation of motion has the form

$$\ddot{y} + \sin y = \varepsilon(\sin \nu t - \lambda \dot{y}).$$

The coordinates of stationary resonant points are found from the equations ($q = 1$)

$$\sqrt{2E_0} = \frac{2\nu}{\pi p} K \left(\sqrt{\frac{2}{E_0}} \right),$$

$$\int_0^{2\pi} \sin \left(\varphi_0 + \nu \int_0^y \frac{d\eta}{\sqrt{2(E_0 - V)}} \right) dy = 4\lambda \sqrt{2E_0} G \left(\sqrt{\frac{2}{E_0}} \right).$$

Here K is the complete elliptic integral of the first kind, G is the complete elliptic integral of the second kind. Let us write out the stability conditions for these regimes:

$$\frac{\partial T}{\partial E_0} \int_0^{2\pi} \cos(\varphi_0 + \nu I_0) dy < 0,$$

$$\lambda - \frac{\nu}{T_0} \int_0^{2\pi} \cos(\varphi_0 + \nu I_0) \frac{\partial I}{\partial E_0} dy + \frac{2\pi p}{T_0} \int_0^{2\pi} \cos(\varphi_0 + \nu I_0) \frac{\partial T^{-1} I}{\partial E_0} dy > 0,$$

where

$$I = \sqrt{\frac{2}{E}} F\left(\frac{y}{2}, \sqrt{\frac{2}{E}}\right), \quad T = \frac{4}{\sqrt{2E}} K\left(\sqrt{\frac{2}{E}}\right),$$

F is an elliptic integral of the first kind. The case of large energies for the mathematical pendulum is easily investigated with the aid of the formulas of § 3.

As a second example, consider the planar motion of a satellite relative to the center of inertia, moving in a central gravitational field along an orbit close to circular, investigated in ⁽³⁾. The equation of motion of the satellite has the form

$$\ddot{y} + 3a^2 \sin y = \varepsilon(4 \sin t + 2 \sin t \dot{y} + 3a^2 \cos t \sin y),$$

where $\varepsilon \ll 1$ is the small eccentricity of the orbit, t is the angular distance of the radius vector of the center of mass from the perigee of the orbit, y is twice the angle between the radius vector of the center of mass and one of the axes of inertia, $a = \text{const}$. The stationary resonant regimes are found from the equations ($q = 1$)

$$\sqrt{2E_0} = \frac{2}{\pi p} K\left(\sqrt{\frac{6a^2}{E_0}}\right), \quad (11)$$

$$\int_0^{2\pi} [2(\sqrt{2(E_0 - V)} + 2) \sin(\varphi_0 + I_0) + 3a^2 \sin y \cos(\varphi_0 + I_0)] dy = 0.$$

The stability conditions of the stationary regimes take the form

$$\frac{\partial T}{\partial E_0} \int_0^{2\pi} [2(\sqrt{2(E_0 - V)} + 2) \cos(\varphi_0 + I_0) - 3a^2 \sin y \sin(\varphi_0 + I_0)] dy < 0,$$

$$\begin{aligned} & \int_0^{2\pi} \left[\left(\frac{2}{\sqrt{2(E_0 - V)}} - 3a^2 \sin y \frac{\partial I}{\partial E_0} \right) \sin(\varphi_0 + I_0) \right. \\ & \quad \left. + 2(\sqrt{2(E_0 - V)} + 2) \frac{\partial I}{\partial E_0} \cos(\varphi_0 + I_0) \right] dy \\ & - 2\pi p \int_0^{2\pi} \frac{\partial T^{-1} I}{\partial E_0} [2(\sqrt{2(E_0 - V)} + 2) \cos(\varphi_0 + I_0) \\ & \quad - 3a^2 \sin y \sin(\varphi_0 + I_0)] dy < 0; \end{aligned} \quad (12)$$

$$T = \frac{4}{\sqrt{2E}} K \left(\sqrt{\frac{6a^2}{E}} \right), \quad I = \sqrt{\frac{2}{E}} F \left(\frac{y}{2}, \sqrt{\frac{6a^2}{E}} \right).$$

(In (11) and (12) the notation of the first example is preserved.)

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Note: Figure translations are in progress. See original paper for figures.

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