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Abstract

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MATHEMATICS

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ON THE FOURIER COEFFICIENTS OF PARABOLIC FORMS

(Presented by Academician I. M. Vinogradov on 7 IV 1964)

Let

$$f(s) = \sum_{n=1}^{\infty} \lambda(n) \exp\left(\frac{2\pi i s n}{N}\right)$$

be a parabolic form of type $\{\Gamma(N), -k\}$; p a prime number; $l = 1, 2, \dots, p-1$; $s = x + iy$; if f, g are parabolic forms of type $\{G, -k\}$, where $G \subset \Gamma(1)$, then

$$(f, g)_G = \iint_{D_G} f(s) \overline{g(s)} y^{k-2} dx dy,$$

where D_G is a fundamental domain for the group G ;

$$f_{l,p}(s) = \sum_{n \equiv l \pmod{p}} \lambda(n) \exp\left(\frac{2\pi i s n}{Np}\right)$$

is a parabolic form of type $\{\Gamma(Np), -k\}$;

$$\beta = 12 \frac{(4\pi)^{k-1} (f_{l,p}(s), f_{l,p}(s))_{\Gamma(Np)}}{[\Gamma(1) : \Gamma(Np)] N^k \Gamma(K+1)},$$

where $[\Gamma(1) : \Gamma(Np)]$ is the index of $\Gamma(Np)$ in $\Gamma(1)$.

Theorem 1. If $M \gg p^2$,

$$\sum_{m \leq M} |\lambda(mp + l)|^2 = \beta M^k + O(M^{k-0.332} p^{k+3.672}). \quad (1)$$

Theorem 2. If $p \ll M^{0.07}$,

$$\sum_{m \leq N} |\lambda(mp + l)|^2 \ll M^k p^{k-1}. \quad (2)$$

The asymptotic formula (1) for $p = l = 1$, with a somewhat better remainder term, was obtained by Rankin ⁽¹⁾. We shall use some of his results. The first results of the type (1), (2), but in a less precise and general form (for eigenfunctions of Hecke operators occurring in the expansion of parabolic forms generated by theta-functions), were obtained by Yu. V. Linnik ⁽²⁾. His method differs from ours. It is interesting to note that from the estimate (2) for $l = 0$ there would follow the still unproved Peterson conjecture ⁽³⁾ on the eigenvalues of Hecke operators.

For the proof of (1) we use Rankin's ingenious device ⁽¹⁾. The function $f_{l,p}(s)$ generates, in the following way, a certain L -function $Z(s)$. Let

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1);$$

put

$$f_V(s) = f_{l,p}(s) \circ V^{-1} = \sum_{n=1}^{\infty} \lambda_{\gamma,\delta}(n) \exp\left(\frac{2\pi i n s}{Np}\right); \quad f_{\gamma,\delta}(s) = \sum_{n=1}^{\infty} \frac{|\lambda_{\gamma,\delta}(n)|^2}{n^s};$$

let χ be a character (mod Np)

$$g(s, \chi) = \sum_{n=1}^{Np-1} \chi(n) f_{0,n}(s) = \sum_{n=1}^{\infty} \frac{d(n, \chi)}{n^s}$$

(if $(m, n) > 1$, $(m, n, Np) = 1$, by $f_{m,n}(s)$ we mean the function $f_{\gamma,\delta}(s)$, for which $\gamma \equiv m$, $\delta \equiv n \pmod{Np}$, $(\gamma, \delta) = 1$).

The L -function we need is defined by the relation

$$Z(s) = g(s + k - 1, \chi) \cdot L(2s, \chi) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}. \quad (3)$$

It is clear that, in order to prove (1), one must know the asymptotics of the sum

$$\sum_{n \leq M} c_n. \quad (4)$$

Using Rankin's results (1), it is easy to see that the function $Z(s)$ is regular in the s -plane except for the point $s = 1$, at which it has a simple pole with residue $kp^{n-k}\varphi(Np)L(2, \chi_0)E(\chi)$, where χ_0 is the principal character (mod Np),

$$E(\chi) = \begin{cases} 1, & \text{if } \chi = \chi_0, \\ 0, & \text{if } \chi \neq \chi_0; \end{cases}$$

further, $Z(s)$ satisfies a functional equation of a rather complicated form.

With the help of lemmas 11,3 (4) and the following Lemma 2, the study of the asymptotics of the sum (4) is reduced to the study of the integral

$$\int_{1.001-iM^{1/3}}^{1.001+iM^{1/3}} \frac{M^s}{s} Z(s) ds; \quad (5)$$

therefore we need information on the behavior of $Z(s)$ to the left of the line $x = 1$, which is contained in Lemma 3.

The proof of the theorems is based on the following lemmas:

Lemma 1.

$$(f_{l,p}(s), f_{l,p}(s))_{\Gamma(Np)} \ll \frac{2}{p-1} [\Gamma(N) : \Gamma(Np)] p^k (f(s), f(s))_{\Gamma(N)}. \quad (6)$$

Lemma 2. Let $\sigma \in \Gamma(1)$,

$$f_{l,p}(s) \circ \sigma = \sum_{n=1}^{\infty} a(n) \exp\left(\frac{2\pi i s n}{Np}\right).$$

Then, for $M \gg p^2$,

$$\sum_{n \leq M} |a(n)|^2 \ll M^k, \quad (7)$$

where the constant occurring in $\ll$ does not depend on p .

Lemma 3. In the strip $-0.001 \ll x \ll 1.001$

$$Z(x + iy) \ll p^{5.004} |y|^{-2x+2.002} \quad (8)$$

as $|y| \rightarrow \infty$ (if $Z(s)$ has a pole at $s = 1$, then a neighborhood of the pole is excluded).

Relying on these lemmas, we obtain

$$\sum_{n \leq N} c_n = k\beta p^{-k} \varphi(Np) L(2, \chi_0) E(\chi) M + O(p^{5.004} M^{0.668}). \quad (9)$$

From this Theorem 1 already follows easily.

Theorem 2 is obtained from (1) with the aid of Lemma 1.

Let us note in conclusion that Theorems 1 and 2 can be sharpened; moreover, they are easily generalized to the case of progressions $qm + l$, $(q, l) = 1$, q composite.

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Note: Figure translations are in progress. See original paper for figures.

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