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Abstract

Full Text

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AN INHOMOGENEOUS RIEMANN BOUNDARY-VALUE PROBLEM WITH INFINITE INDEX

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MATHEMATICS

1°. Let, in the complex z -plane, a domain D be given whose boundary L is the ray $1 \leq t \leq \infty$. We consider in the domain D the Riemann boundary-value problem with infinite index

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t) \quad (1)$$

under the following assumptions:

$$1) \arg G(t) = 2\pi\varphi(t)t^\rho, \quad -1 < \varphi(1) \leq 0, \quad \varphi(\infty) \neq 0, \quad 0 < \rho < \infty, \quad (2)$$

where $\varphi(t) \in H(\mu)$, i.e.

$$|\varphi(t_1) - \varphi(t_2)| < A \left| \frac{1}{t_1} - \frac{1}{t_2} \right|^\mu, \quad t_k \in L, \quad A, \mu = \text{const}, \quad (3)$$

and

$$\frac{\rho}{\rho+1} < \mu \leq 1; \quad (4)$$

$$2) \ln |G(t)|, \quad g(t) \in H(\lambda) \quad (0 < \lambda \leq 1); \quad (5)$$

$$3) \quad g(1) = g(\infty) = 0. \quad (6)$$

The case $g(t) \equiv 0$ (the homogeneous problem) was considered in ^(5,6). The inhomogeneous problem was solved in ⁽⁵⁾ for the case $0 < \rho < 1/2$, $\mu > \rho$, $\varphi(\infty) > 0$.

2°. **Definition 1.** Let a set of points $\{z_n\}$ be given in the domain D , with $\lim_{n \rightarrow \infty} z_n = \infty$. Then, if for all ψ_1, ψ_2 such that

$$-2\pi < \psi_1 < \psi_2 < 2\pi, \quad |\psi_1 - \psi_2| < 2\pi, \quad \psi_k \in N,$$

where N is at most countable, for some $\sigma > 0$ there exists the limit

$$\lim_{r \rightarrow \infty} \frac{1}{r^\sigma} \sum_{\substack{\psi_1 < \arg z_n \leq \psi_2 \\ |z_n| \leq r}} \arg z_n = \nu(\psi_1, \psi_2) \neq \infty \quad (7)$$

and the asymptotic estimate

$$\sum_{\substack{-\pi < \arg z_n \leq \pi \\ |z_n| \leq r}} |\arg z_n| < K r^\sigma \quad (K > 0, K = \text{const})$$

holds, then we shall say that the set $\{z_n\}$ has, in the domain D , an **argument density**. In doing so we adopt, by definition,

$$\nu(\psi, \psi) = 0; \quad \nu(\psi_1, \psi_2) = -\nu(\psi_2, \psi_1), \quad \text{if } \psi_1 > \psi_2,$$

$$\nu(\psi_1, \psi_2) = \nu(\psi_1, +0) + \nu(+0, \psi_2), \quad \text{if } |\psi_1 - \psi_2| > 2\pi.$$

By the **argument density** we shall mean the function $\nu(\psi) = \nu(\psi_0, \psi)$, determined up to an additive constant for fixed ψ_0 , extended at the points of the set N by right-continuity.

The concept of argument density is closely connected with the concept of angular density introduced by B. Ya. Levin ((²), p. 118). We note that the quantity $\sum_{|z_n| \leq r} \sin \arg z_n$, related to the sum in (7), was considered by R. Nevanlinna ((³)).

We shall call the following function the **indicator** of a function $f(z)$, regular and of order $\sigma > 0$ in the angle $\alpha < \arg z < \beta$ ((⁵), p. 1247; (⁴), p. 209):

$$h_f(\theta) = \overline{\lim}_{r \rightarrow \infty} r^{-\sigma} \ln |f(re^{i\theta})| \quad (\alpha < \theta < \beta).$$

Definition 2. A function $f(z)$, regular and of order $\sigma > 0$ in the angle $\alpha < \arg z < \beta$, continuous for $\alpha \leq \arg z \leq \beta$ (for $z \neq \infty$), is called a function of **completely regular growth in the closed angle** $\alpha \leq \arg z \leq \beta$, if the function $r^{-\sigma} \ln |f(re^{i\theta})|$ tends to $h_f(\theta)$ uniformly for $\alpha \leq \theta \leq \beta$, as r tends to $+\infty$, except for values from a set E , common to all θ , of zero relative measure ((²), pp. 127, 182).

We shall call $f(z)$ a function of **completely regular growth** (of order $\sigma > 0$) in the **open angle** $\alpha < \arg z < \beta$, if it has completely regular growth in each angle $\alpha + \varepsilon \leq \arg z \leq \beta - \varepsilon$ ($\varepsilon > 0$) and if, asymptotically,

$$\sup_{|z| \leq r} |f(z)| < \exp(Kr^\sigma) \quad (K > 0).$$

Denote by B_σ the class of functions regular, of finite order $\sigma > 0$, and of completely regular growth in the domain D .

3°. Consider problem (1) for the case of a plus-infinite index, i.e., when $\varphi(\infty) > 0$. We shall solve it in the class B_σ , where $0 < \sigma < \min(\rho, \frac{1}{2})$. (For $\sigma > \min(\rho, \frac{1}{2})$ there are no solutions.) Denote by $\Psi(z)$ the solution of the corresponding homogeneous problem (1):

$$\Psi^+(t) = -G(t)\Psi^-(t). \quad (8)$$

Theorem 1. *Let $\Phi_0(z)$ be some bounded solution of problem (1) (with $\varphi(\infty) > 0$) of order not less than σ_0 , $0 < \sigma < \sigma_0 < \min(\rho, \frac{1}{2})$. Then the general solution of this problem in the class B_σ has the form*

$$\Phi(z) = \Phi_0(z) + \Psi(z), \quad (9)$$

where $\Psi(z)$ is the general solution of the homogeneous problem (8) in the class B_σ . Moreover, the indicators of the functions $\Phi(z)$ and $\Psi(z)$ in the interval $0 < \theta < 2\pi$ coincide.

The general solution of the homogeneous problem was found in ⁽⁶⁾. It remains to find $\Phi_0(z)$. Applying the method of F. D. Gakhov (⁽¹⁾, p. 117), one can obtain:

$$\Phi_0(z) = \frac{\Psi_0(z)}{2\pi i} \int_1^\infty \frac{g(x) dx}{\Psi_0^+(x)(x-z)}, \quad (10)$$

where $\Psi_0(z)$ is some solution of the homogeneous problem (8). In order that the integral converge and that $\Phi_0(z)$ be bounded and of order not less than σ_0 , it is sufficient to find such a $\Psi_0(z)$ that the following conditions be satisfied:

- a) $\Psi_0(z)$ is bounded in D ;
- b) in any angle $\varepsilon \leq \arg z \leq 2\pi - \varepsilon$, for $|z| \equiv r > r_\varepsilon$, the estimate

$$\max_{\varepsilon \leq \theta \leq 2\pi - \varepsilon} |\Psi_0(re^{i\theta})| < \exp(-K_\varepsilon r^{\sigma_0}) \quad (K_\varepsilon > 0);$$

- c) $g(t)/\Psi_0^+(t) \in H(\tau) \quad (0 < \tau \leq 1), \lim_{t \rightarrow \infty} g(t)/\Psi_0^+(t) = 0$.

One of the simplest functions subject to conditions a)–c) has the form*

$$\Psi_0(z) = \exp \left[\frac{z}{2\pi i} \int_0^\infty \frac{\ln G(x) - 2\pi i n_{\Psi_0}(x)}{x(x-z)} dx \right] \prod_{n=1}^\infty \frac{\left(1 - \frac{z}{r_n} e^{-ir_n^{-\rho}}\right) \left(1 + \frac{z}{s_n}\right)}{\left(1 - \frac{z}{r_n}\right) \left(1 - \frac{z}{s_n}\right)}, \quad (11)$$

where

$$s_n = \left(\frac{2n-1}{2 \cos \sigma_0 \pi} \right)^{1/\sigma_0}, \quad 1 \leq r_1 \leq r_2 \leq \dots,$$

and the number $n_1(r)$ of points r_n on the interval $[1, r]$ is determined by the equality

$$n_1(r) = \max \left\{ \left[\max_{1 \leq x \leq r} \left\{ \varphi(x)x^\rho - x^{\sigma_0} + \frac{1}{2} \right\} \right], 0 \right\}^{**},$$

while $n_{\Psi_0}(r)$ is the total number of points $s_n, r_n \in [0, r]$ ((6), p. 16).

Let us formulate the final result.

Theorem 2. The general solution of the nonhomogeneous problem (1) in the class B_σ , $0 < \sigma < \min(\rho, 1/2)$, has the form

$$\Phi(z) = \frac{\Psi_0(z)}{2\pi i} \int_1^\infty \frac{g(x) dx}{\Psi_0^+(x)(x-z)} + cz^m \exp \left[\frac{z}{2\pi i} \int_0^\infty \frac{\ln G(x) - 2\pi i \tilde{n}(x)}{x(x-z)} dx \right] \times \prod_{n=1}^\infty \frac{1 - z/z_n}{1 - z/|z_n|}, \quad (12)$$

where $\Psi_0(z)$ is defined by formula (11), $\tilde{n}(r)$ is the number of points z_n in $0 < |z| \leq r$, and the following conditions are satisfied:

I. The set $\{z_n\}$ has angular σ -density $\nu(\psi)$.

II. There exists the finite limit

$$\lim_{r \rightarrow \infty} \frac{\sigma^2}{r^\sigma} \int_0^r \frac{dt}{t} \int_0^t \frac{\frac{1}{2\pi} \arg G(x) - \tilde{n}(x)}{x} dx = \gamma \geq 0,$$

where, if $\gamma = \gamma(\sigma) = 0$, then $\gamma(\sigma - \varepsilon) = +\infty$ for any $\varepsilon > 0$.

III. The integral

$$\int_0^\infty \frac{1}{x^2} \left[\frac{1}{2\pi} \arg G(x) - \tilde{n}(x) \right] dx$$

converges.

IV. For $t > t_\infty$, $t \neq |z_n|$, the estimate

$$t \int_0^\infty \frac{\frac{1}{2\pi} \arg G(x) - \tilde{n}(x)}{x(x-t)} dx + m \ln t + \sum_{n=1}^\infty \ln \left| \frac{z_n - t}{|z_n| - t} \right| < C_\Phi = \text{const}$$

holds.

In this case the indicator of the solution is expressed by the formula ($0 < \theta < 2\pi$)

$$h_{\Phi}(\theta) = \frac{\pi}{\sin \pi \sigma} \left[\int_{-\pi}^{\pi} \alpha(\psi, \theta) d\nu(\psi) - \gamma \cos \sigma(\theta - \pi) \right],$$

where it is set that

$$\alpha(\psi, \theta) = \begin{cases} \psi^{-1} [\cos \sigma(|\theta - \psi| - \pi) - \cos \sigma(\theta - \pi)], & 0 < \psi \leq \pi, \\ \psi^{-1} [\cos \sigma(|\theta - \psi - 2\pi| - \pi) - \cos \sigma(\theta - \pi)], & -\pi \leq \psi < 0, \\ \sigma \sin \sigma(\theta - \pi), & \psi = 0. \end{cases}$$

Corollary. Problem (1) has in the class B_{σ} , $0 < \sigma < \min(\rho, 1/2)$, an infinite set of linearly independent solutions.

* We assume that $G(t) = 1$, $g(t) = 0$ for $0 \leq t < 1$.

** $[a]$ denotes the integer part of the real number a .

Remark. The convergence of the infinite product and of the second of the integrals in equality (12) follows respectively from I and III.

4°. Let us now consider problem (1) for minus-infinite index.

Theorem 3. Let assumptions (2)–(6) be satisfied, with $\varphi(\infty) < 0$, and let $\tilde{\Psi}(z)$ be meromorphic in the domain D and such that:

- 1) $\tilde{\Psi}^+(t) = G(t)\tilde{\Psi}^-(t)$ ($1 < t < \infty$);
- 2) $g(t)/\tilde{\Psi}^+(t) \in H$;
- 3) $|\tilde{\Psi}^{\pm}(t)| < \text{const}$ ($1 \leq t < \infty$);
- 4) on some sequence of circles $|z| = k_n$ ($k_n \rightarrow \infty$)

$$\frac{1}{M} < \sup_{n=1,2,\dots} \max_{|z|=k_n} |\tilde{\Psi}(z)| < M < \infty,$$

where $M = \text{const}$.

Then, for the solvability of problem (1) in the class of bounded functions, it is necessary and sufficient that at all poles z_n of the function $\tilde{\Psi}(z)$, the number of which is known to be infinite, the equalities

$$\int_1^{\infty} \frac{g(x) dx}{\tilde{\Psi}^+(x)(x - z_n)} = 0 \quad (n = 1, 2, \dots) \quad (13)$$

hold.

If (13) is fulfilled, the solution is unique and is expressed by the formula

$$\Phi(z) = \frac{\tilde{\Psi}(z)}{2\pi i} \int_1^\infty \frac{g(x) dx}{\tilde{\Psi}^+(x)(x-z)}. \quad (14)$$

One of the simplest functions subject to conditions 1)–4) has the form

$$\tilde{\Psi}(z) = \exp \left[\frac{z}{2\pi i} \int_1^\infty \frac{\ln G(x) + 2\pi i p(x)}{x(x-z)} dx \right] \prod_{n=1}^\infty \frac{1-z/|z_n|}{1-z/z_n},$$

where $p(r)$ denotes the number of poles z_n of the function $\tilde{\Psi}(z)$ in $0 < |z| \leq r$, and

$$z_n = r_n e^{i r_n^{-\rho}}, \quad p(r) = \left[\max_{1 \leq x \leq r} \left\{ \frac{1}{2} - \varphi(x) x^\rho \right\} \right].$$

Remark. For $0 < \rho < 1/2$, (14) can be simplified to the form

$$\Phi(z) = \frac{X(z)}{2\pi i} \int_1^\infty \frac{g(x) dx}{X^+(x)(x-z)}, \quad \text{where} \quad X(z) = \exp \left[\frac{z}{2\pi i} \int_1^\infty \frac{\ln G(x) dx}{x(x-z)} \right]$$

(if (13) is fulfilled); in this case the integrals are certainly convergent.

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CITED LITERATURE

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