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Abstract

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MATHEMATICS

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TENSOR FIELDS ON COMPACT HOMOGENEOUS SPACES ADMITTING A SPHERICAL MODEL

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Let there be given a homogeneous space $K = G/H$, where $G \supset H$ are Lie groups, G is connected, and H is its closed subgroup. Suppose that we have constructed a faithful representation of the group G in the form of a group of motions $\varphi(G)$ of the unit sphere S_n of the Euclidean space E_{n+1} , admitting such a closed surface of transitivity K^* that the stationary subgroup H of the group G is mapped into the stationary subgroup $\varphi(H)$ of the group $\varphi(G)$, considered as a group of transformations on K^* , and at the same time the representation $\bar{\varphi}(G)$ by a group of motions on K^* is faithful. In this case we shall say that K^* is a spherical model of the homogeneous space K .

It is clear that, in order for a homogeneous space K to admit a spherical model, it is necessary that the group G , and consequently also K , be compact. Conversely, in view of the compactness of G , it has a faithful representation by orthogonal matrices, and moreover the compact subgroup H is singled out algebraically in this representation; then one can construct a spherical model of the homogeneous space K in the same way as in ⁽¹⁾ (p. 198) the projective model of the homogeneous space K is constructed.

In ⁽²⁾ a description was given of the spaces of tensor fields on the n -dimensional sphere S_n , irreducible with respect to the group of motions $O(n+1)$ (which may be regarded as the homogeneous space $K = O(n+1)/O(n)$).

The purpose of the present paper is, first, to give a scheme for obtaining analogous results in the general case, when $K = G/H$ is a homogeneous space admitting a spherical model; and, second, to carry this out explicitly for the Grassmann manifold $K = O(m)/O(m-k)O(k)$.

The induced tensor field $kT_{\alpha_1, \dots, \alpha_l}$, introduced in ⁽²⁾, may be interpreted as the projection onto the sphere S_n of a constant tensor field $T_{i_1, \dots, i_k}$ of the space E_{n+1} , with the annihilated indices being projected onto the normal to S_n , and the remaining ones onto the plane tangent to S_n .

Let us take a spherical model K^* of our homogeneous space K on the unit sphere S_n of the Euclidean space E_{n+1} . We decompose the set of all possible tensors in E_{n+1} into linear spaces Π , irreducible with respect to the group $\varphi(G)$. Just as in ⁽²⁾, we project, in all the indicated ways, these tensor spaces Π , irreducible with respect to $\varphi(G)$, onto the sphere S_n . We obtain, as in ⁽²⁾, a complete system of spaces Π' of tensor fields on S_n , irreducible with respect to the group $\varphi(G)$ (complete in the sense that every continuous tensor field on S_n admits uniform approximation by finite sums of our induced tensor fields).

Now we project the tensor fields obtained on S_n onto our model K^* . In doing so, we project all indices onto the plane tangent to K^* . The tensor fields on K^* obtained in this way will likewise be called induced.

Under the projection of an irreducible space Π' of tensor fields on S_n onto K^* , the preimage of the zero tensor field on K^* on S^* either fills the whole space Π' , or coincides with the zero tensor field on S_n . In the latter case the mapping under projection will be one-to-one, and the resulting space Π^* of tensor fields on K^* will be irreducible.

The set of irreducible spaces Π^* of induced tensor fields on K^* is complete in the sense that every continuous tensor field on K^* admits a uniform approximation by finite sums of the given induced tensor fields.

Indeed, every continuous tensor field P^* on K^* can be represented as the projection of a tensor field P' , continuous on S_n . If a sequence of tensor fields on S_n converges uniformly to the tensor field P' , then the sequence of projections on K^* of these tensor fields also converges uniformly to the projection P^* of the tensor field P' . Hence our assertion follows from Theorem 1 ⁽²⁾.

With this we finish the consideration of the general case and pass to Grassmann manifolds.

Thus,

$$K = O(m)/O(m-k)O(k),$$

where $O(m)$ is the proper orthogonal group.

As is known, the subgroup $O(m-k)O(k)$ is singled out from the group $O(m)$ by the condition of invariance of a simple k -vector (or of an $(m-k)$ -vector orthogonal to it).

For E_{n+1}^1 , where

$$n+1 = \frac{m!}{(m-k)!k!},$$

we take the Euclidean space of k -vectors $z^{i_1, \dots, i_k}$ with metric

$$(z_i, z_j) = g_{i_1, j_1} g_{i_2, j_2} \cdots g_{i_k, j_k} z^{i_1, i_2, \dots, i_k} z^{j_1, j_2, \dots, j_k}.$$

The space K^* will be the intersection of the manifold of simple k -vectors with the unit sphere S_n .

From a tensor T of valence ks of the space E_m we can obtain a tensor $\hat{T}$ in the space E_{n+1} in the following way: the indices of the tensor T are divided into s groups of k indices in each, and alternation is carried out within each group; each such group is taken as an index in the space of k -vectors. We thus obtain in E_{n+1} a tensor $\hat{T}$ of valence s (provided only that $\hat{T} \neq 0$). Such a tensor in E_{n+1} will be called **induced**. It can be shown, relying on tensors having only one coordinate different from zero, that every tensor in E_{n+1} is representable in the form of a sum of induced tensors.

If, for tensors T in E_m belonging to an irreducible space Π , the indices are uniformly divided into groups, then the corresponding space $\hat{\Pi}$ of tensors $\hat{T}$ in E_{n+1} will be irreducible with respect to the group $O(m)$, acting in the obvious way in E_{n+1} . The correspondence between tensors of the spaces Π and $\hat{\Pi}$ in each case will either be one-to-one, or the space $\hat{\Pi}$ will be zero.

Proposition 1. *If m and k are even, then on K there acts not $O(m)$, but its factor by the symmetry about the origin, and in the space of tensor fields T^* on K^* there occur precisely those irreducible representations of the group $O(m)$ which are equivalent to representations by means of irreducible tensors T in E_m with an even number of indices. If, however, at least one of the numbers m and k is odd, then every finite-dimensional irreducible representation of the group has an equivalent one among the irreducible representations in the space of tensor fields on*

$$K = O(m)/O(m-k)O(k).$$

The space of all tensors in E_m decomposes into subspaces irreducible with respect to the group $O(m)$, the tensors of each of which are representable in the form of a product of some number s of metric tensors by a traceless tensor with a Young scheme ⁽³⁾.

For each such space the Young scheme and the number s of metric tensors are fixed. The set of such spaces Π of tensors T in E_m we shall call M . It is complete in the sense that every tensor in E_m is representable as a finite sum of tensors, each of which belongs to some irreducible space—an element of M . If, for each element of M , the indices of the tensors belonging to it are divided into groups of k indices in each (if this is possible) in all possible ways and, after alternating over these groups, one obtains spaces of tensors in E_{n+1} irreducible with respect to the group $\varphi(O(m))$, then the resulting set-

into M' of these spaces is complete in the preceding sense. In contrast to (3), we shall take the Young diagrams in skew-symmetric form.

It is of interest to note certain restrictions on the arrangement of groups of indices with respect to k , in spite of which the obtained set M' preserves completeness.

Proposition 2. *In order to preserve the completeness of the set M' , it is sufficient to take only those partitions into groups for which no group contains*

two indices from one row of the Young diagram or two indices of one metric tensor.

From the set M' we can, by the method described above, obtain a set M^* of spaces of tensor fields on K^* , irreducible with respect to the group $\varphi(O(m))$, complete in the sense that any continuous tensor field on K^* admits a uniform approximation by finite sums of continuous tensor fields, each of which belongs to an irreducible space—an element of M^* .

Moreover, it is of interest to consider also restrictions which, perhaps, destroy the completeness of M' , but preserve the completeness of M^* .

Theorem. *Without violating the completeness of the set M^* , among all possible ways of partitioning the indices of an element of $\widehat{M}$ into groups one may take only those such that: 1) they satisfy Proposition 2; 2) each group, when projected (as an index of a tensor in E_{n+1}) subsequently onto the normal to S_n , has no indices in rows of the Young diagram with number greater than k ; 3) each group of indices which, under projection onto S_n , is projected onto the tangent plane has no more than one index from rows of the Young diagram with number greater than k .*

It follows from this that the number of indices in rows with number greater than k must not exceed $\frac{1}{k}$ of the total number of indices of the tensor.

The system of restrictions obtained is, unfortunately, not yet complete. For example, for $K = O(m)/O(m-2)O(2)$ the tensor with Young diagram

$$\begin{array}{cc} 1 & 1 \\ 3 & 4 \\ 5 & 6 \end{array}$$

with groups of indices 1, 4; 3, 6; 2, 5 satisfies all the conditions, but already under the mapping into M' a zero element is obtained.

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Note: Figure translations are in progress. See original paper for figures.

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