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Abstract

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MATHEMATICS

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ON THE MAGNITUDES OF THE DEFECTS OF A MEROMORPHIC FUNCTION

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§ 1. Let $f(z)$ be a function meromorphic in the open plane; $T(r, f)$, $m(r, a)$, $n(r, a)$, $N(r, a)$, $\delta(a, f)$ are the quantities introduced by R. Nevanlinna, characterizing the distribution of the values of this function. We agree to denote

$$n(r) = n(r, 0) + n(r, \infty), \quad N(r) = N(r, 0) + N(r, \infty),$$

by the letter K with indices absolute constants, and by the letter C with indices quantities depending only on the function under consideration.

As is known ⁽¹⁾, the defect quantities satisfy the relation

$$\sum_{(a)} \delta(a) \leq 2$$

(the sum is extended over all values a with $\delta(a, f) > 0$). Put

$$s(f) = \sum_{(a)} \sqrt{\delta(a)}, \quad \sigma(\lambda) = \sup_{(f)} s(f),$$

where the supremum is taken over all meromorphic functions of lower order λ .

B. Fuchs ⁽²⁾ proved that for $\lambda < \infty$ the quantity $\sigma(\lambda)$ is finite and that the estimate

$$\sigma(\lambda) \leq K_1 (1 + \sqrt{\lambda |\ln \lambda|}). \quad (1,1)$$

is valid.

The main result of this article is the following theorem.

Theorem 1. The inequality

$$\sigma(\lambda) \leq K_2 \sqrt{\lambda}, \quad 0.5 \leq \lambda < \infty. \quad (1,2)$$

is valid.

For the entire function $((^1), p.240)$

$$h_p = \int_0^z e^{t^p} dt, \quad p = 1, 2, 3, \dots,$$

one has $\lambda = p$,

$$s(h_p) = 1 + \sqrt{p} = 1 + \sqrt{\lambda}.$$

Therefore estimate (1,2) is sharp for large λ in the sense of order.

We obtain Theorem 1 as a consequence of two theorems: Theorem 2, due to Fuchs $(^2)$, and Theorem 3, established by us.

Theorem 2 $(^2)$. If $f(z)$ is a meromorphic function of finite lower order λ and has at least two deficient values, then the relation

$$\sum_{(a)} \sqrt{\delta(a)} \leq \left\{ 2\pi \lim_{r \rightarrow \infty} [T(r, f)]^{-1} r \mathfrak{M} \left(r, \frac{f''}{f'} \right) \right\}^{1/2}, \quad (1,3)$$

is valid, where

$$\mathfrak{M}(r, g) = \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})| d\theta.$$

Theorem 3. If $f(z)$ is a meromorphic function of lower order λ ($\lambda \geq 0.5$), then

$$\lim_{r \rightarrow \infty} [T(r, f)]^{-1} r \mathfrak{M} \left(r, \frac{f''}{f'} \right) \leq K_3 \lambda. \quad (1,4)$$

§ 2. Auxiliary propositions.

Lemma 1. Let $f(z)$ be a function meromorphic in the sector

$$G_{a,R,\vartheta} = \{z : 0 < |z| < R, |\arg z - \vartheta| < a\}.$$

For any real ϑ , any α ($0 < \alpha < \pi$), and r ($0 < r_0 < r < 0.5R$), the inequality holds

$$\begin{aligned}
\frac{r}{2\pi} \int_{-\alpha/2}^{\alpha/2} \left| \frac{f'(re^{i(\varphi+\vartheta)})}{f(re^{i(\varphi+\vartheta)})} \right| d\varphi &\leq \alpha^{-1} \int_{r_0}^R \{ |\ln |f(te^{i(\vartheta+\alpha)})|| + |\ln |f(te^{i(\vartheta-\alpha)})|| \} P(t, r, \alpha) dt \\
&+ K_4 \alpha^{-1} \int_0^R \{ |\ln |f(te^{i(\vartheta+\alpha)})|| + |\ln |f(te^{i(\vartheta-\alpha)})|| \} \left(\frac{r}{R^2} \right)^x t^{x-1} dt \\
&+ K_5 \alpha^{-1} \int_{-\alpha}^{\alpha} |\ln |f(Re^{i(\vartheta+\theta)})|| \left(\frac{r}{R} \right)^x d\theta \\
&+ 2 \sum_{c_m \in G_{\alpha, R, 0}} \left(\frac{r}{|c_m|} \right)^x \Phi \left[\left(\frac{r}{|c_m|} \right)^x \right] \\
&+ K_6 \sum_{c_m \in G_{\alpha, R, 0}} \left(\frac{r}{R^2} \right)^x |c_m|^x + C_1,
\end{aligned} \tag{2.1}$$

where

$$x = x(\alpha) = \pi(2\alpha)^{-1}, \quad P(t, r, \alpha) = t^{x-1} r^x (t^x + r^x)^{-1}, \quad \Phi(u) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|ue^{i\theta} - 1|},$$

$c_m = c_m(\vartheta)$ are the zeros and poles of the meromorphic function $f(ze^{i\vartheta})$.

The proof is based on the representation of $\ln f(re^{i\theta})$ in the sector $G_{\alpha, R, 0}$, analogous to the Schwarz–Nevanlinna formula ((1), p. 165).

Lemma 2. Let $f(z)$ be a function meromorphic in $|z| \leq R < \infty$. For $0 < r_0 < r < 0.5R$, $0 < \alpha < \pi$, the estimate holds

$$\begin{aligned}
rm \left(r, \frac{f'}{f} \right) &\leq K_7 \alpha^{-2} \int_{r_0}^R \{ m(t, 0) + m(t, \infty) \} P(t, r, \alpha) dt \\
&+ K_8 \alpha^{-1} \left(\frac{r}{R} \right)^x T(R, f) + K_9 \sum_{|c_m| < R} \left(\frac{r}{|c_m|} \right)^x \Phi \left[\left(\frac{r}{|c_m|} \right)^x \right] \\
&+ K_{10} \left(\frac{r}{R} \right)^x n(R) + C_2 \alpha^{-1}.
\end{aligned} \tag{2.2}$$

Proof. Let $\vartheta_k = \beta + k\alpha$, where $0 \leq \beta < 2\pi$, and k takes the values $0, 1, \dots, q = [4x]$. Putting $\vartheta = \vartheta_k$ ($k = 0, 1, \dots, q$) in inequality (2.1), we obtain $q + 1$ inequalities. Adding these inequalities over k from 0 to q , we shall have

$$\begin{aligned}
rm\left(r, \frac{f'}{f}\right) &\leq \alpha^{-1} \sum_{k=0}^q \int_{r_0}^R \{|\ln |f(te^{i(\beta+(k+1)\alpha)})|| + |\ln |f(te^{i(\beta+(k-1)\alpha)})||\} P(t, r, \alpha) dt \\
&\quad + K_4 \alpha^{-1} \sum_{k=0}^q \{|\ln |f(te^{i(\beta+(k+1)\alpha)})|| + |\ln |f(te^{i(\beta+(k-1)\alpha)})||\} \left(\frac{r}{R^2}\right)^x t^{x-1} dt \\
&\quad + K_5 \alpha^{-1} \sum_{k=0}^q \int_{-\alpha}^{\alpha} |\ln |f(Re^{i(\theta+\beta+k\alpha)})|| \left(\frac{r}{R}\right)^x d\theta \\
&\quad + 2 \sum_{k=0}^q \sum_{c_m(\vartheta_k) \in G_{\alpha, R, 0}} \left(\frac{r}{|c_m|}\right)^x \Phi\left[\left(\frac{r}{|c_m|}\right)^x\right] \\
&\quad + K_6 \sum_{k=0}^q \sum_{c_m(\vartheta_k) \in G_{\alpha, R, 0}} \left(\frac{r}{R^2}\right)^x |c_m|^x C_2 \alpha^{-1}.
\end{aligned} \tag{2.3}$$

Obviously, the inequalities hold

$$2 \sum_{k=0}^q \sum_{c_m(\vartheta_k) \in G_{\alpha, R, 0}} \left(\frac{r}{|c_m|}\right)^x \Phi\left[\left(\frac{r}{|c_m|}\right)^x\right] \leq K_9 \sum_{|c_m| \leq R} \left(\frac{r}{|c_m|}\right)^x \Phi\left[\left(\frac{r}{|c_m|}\right)^x\right], \tag{2.4}$$

$$\sum_{k=0}^q \int_{-\alpha}^{\alpha} |\ln |f(Re^{i(\theta+\beta+k\alpha)})|| d\theta \leq K_8 T(R, f) \tag{2.5}$$

$$\sum_{k=0}^q \sum_{c_m(\vartheta_k) \in G_{\alpha, R, 0}} \left(\frac{r}{R^2}\right)^x |c_m|^x \leq K_{11} \sum_{|c_m| \leq R} \left(\frac{r}{R^2}\right)^x |c_m|^x. \tag{2.6}$$

Replacing in inequality (2,3) the expressions occurring on the left-hand sides of (2,4), (2,5), and (2,6) by the expressions occurring on the right-hand sides, and integrating the resulting inequality with respect to β from 0 to 2π , we obtain inequality (2,2).

§ 3. Proof of Theorem 3. Let $f(z)$ have lower order λ and order ρ . We shall carry out the proof under the assumption that $\lambda < \rho$. Choose γ so that $\lambda < \gamma < \rho$, and take $\alpha < \pi(2\gamma)^{-1}$. Divide inequality (2,2) by $r^{\gamma+1}$ and integrate it with respect to r from r_0 to $0.5R$; we obtain

$$\begin{aligned}
\int_{r_0}^{0.5R} r^{-\gamma-1} \left\{ r \mathfrak{M} \left(r, \frac{f'}{f} \right) \right\} dr &\leq K_7 \alpha^{-2} \int_{r_0}^{0.5R} \{m(t, 0) + m(t, \infty)\} \times \\
&\times \int_{r_0}^{0.5R} r^{-\gamma-1} P(t, r, \alpha) dr dt + K_7 \alpha^{-2} \int_{0.5R}^R \{m(t, 0) + m(t, \infty)\} \times \\
&\times \int_{r_0}^{0.5R} r^{-\gamma-1} P(t, r, \alpha) dr dt + K_{11} (\pi - 2\alpha\gamma)^{-1} R^{-\gamma} T(R, t) + \\
&+ K_9 \sum_{r_0 < |c_m| < 0.5R} \int_{r_0}^{0.5R} r^{-\gamma-1} \left\{ \left(\frac{r}{|c_m|} \right)^{\chi} \Phi \left[\left(\frac{r}{|c_m|} \right)^{\chi} \right] \right\} dr \\
&+ K_{12} (\pi - 2\alpha\gamma)^{-1} R^{-\gamma} n(R) + \\
&+ K_9 \sum_{0.5R \leq |c_m| < R} \int_{r_0}^{0.5R} r^{-\gamma-1} \left\{ \left(\frac{r}{|c_m|} \right)^{\chi} \Phi \left[\left(\frac{r}{|c_m|} \right)^{\chi} \right] \right\} dr + C_4 \alpha^{-1}.
\end{aligned} \tag{3.1}$$

Using the relations (see ^(10, 3))

$$\int_0^\infty r^{-\gamma-1} P(t, r, \alpha) dr = t^{-\gamma-1} \alpha \sec \alpha \gamma, \quad \int_0^\infty r^{-\sigma} \Phi(r) dr \leq 4.4 \operatorname{cosec} \pi \sigma \quad (0 < \sigma < 1), \tag{3.2}$$

from (3.1) we obtain

$$\begin{aligned}
\sin 2\alpha\gamma \int_{r_0}^{0.5R} r^{-\gamma-1} \left\{ r \mathfrak{M} \left(r, \frac{f'}{f} \right) \right\} dr &\leq 2K_7 \alpha^{-1} \sin \alpha\gamma \int_{r_0}^{0.5R} r^{-\gamma-1} \{m(r, 0) + m(r, \infty)\} dr + \\
&+ 2K_{13} \alpha^{-1} \sin \alpha\gamma R^{-\gamma} T(R, t) + K_{11} (\pi - 2\alpha\gamma)^{-1} \sin 2\alpha\gamma R^{-\gamma} T(R, f) + \\
&+ K_{14} \alpha \gamma^2 \int_{r_0}^{0.5R} r^{-\gamma-1} N(r) dr + C_5 R^{-\gamma} T(2R, f) + C_6.
\end{aligned} \tag{3.3}$$

Choose in this inequality $\alpha = \pi(4\gamma)^{-1}$. Applying it then to $f'(z)$ instead of $f(z)$ (this can be done, since the order and lower order of $f(z)$ and $f'(z)$ coincide) (⁽⁴⁾, p. 52) and taking into account the relation (⁽⁵⁾, p. 61)

$$T(r, f') \leq 2T(r, f) + 4 \ln^+ T(2r, f) + 4 \ln^+ r + K_{15} \quad (0 < r_0 < r),$$

from (3.3) we find

$$\int_{r_0}^{0.5R} r^{-\gamma-1} \left\{ r \mathfrak{M} \left(r, \frac{f''}{f'} \right) \right\} dr \leq K_{16} \gamma \int_{r_0}^{0.5R} r^{-\gamma-1} T(r, f) dr +$$

$$+C_7 \int_{r_0}^{0.5R} r^{-\gamma-1} \ln^+ T(r, f) dr + C_8 R^{-\gamma} \{T(4R, f) + \ln^+ T(4R, f)\} + C_8.$$

From this inequality, by means of arguments analogous to those used in ^(9, 7, 10), we obtain

$$\lim_{r \rightarrow \infty} [T(r, f)]^{-1} \left\{ r \mathfrak{M} \left(r, \frac{f''}{f'} \right) \right\} \leq K_{16} \gamma.$$

Letting now γ tend to λ , we obtain the assertion of the theorem.

For $\lambda = \rho$ the proof of Theorem 3 is carried out as follows. Divide inequality (2.2) with $\alpha = \pi(4\rho)^{-1}$ by $r^{\rho(r)+1}$, where $\rho(r)$ is the refined order of the meromorphic function $f(z)$, and integrate with respect to r from r_0 to $0.5R$. Arguing further analogously to the case $\lambda < \rho$, but in estimating the integrals (3.2) using Lemma 2 from ⁽⁶⁾ (p. 78), we arrive at the relation

$$\begin{aligned} & \int_{r_0}^{0.5R} r^{-\rho(r)-1} \left\{ r \mathfrak{M} \left(r, \frac{f''}{f'} \right) \right\} dr \leq K_{18} \rho \int_{r_0}^{0.5R} r^{-\rho(r)-1} T(r, f) dr + \\ & + C_{10} \int_{r_0}^{0.5R} r^{-\rho(r)-1} \ln^+ T(r, f) dr + C_{11} R^{-\rho(R)} \{T(4R, f) + \ln^+ T(4R, f)\} + C_{12}. \end{aligned}$$

Taking into account the properties of the refined order, we have

$$\lim_{R \rightarrow \infty} R^{-\rho(R)} T(R, f) = 1, \quad \lim_{R \rightarrow \infty} \int_{r_0}^{0.5R} r^{-\rho(r)-1} T(r, f) dr = \infty.$$

Using these relations and arguing analogously (cf. ^(9, 7, 10)) to the case $\lambda < \rho$, we obtain the assertion of the theorem.

§ 4. Theorem 4. Let Δ be the sum of the deficiencies of a meromorphic function $f(z)$ of finite lower order λ . Then $f(z)$ has at least one deficiency satisfying the condition

$$\delta(a) > \Delta^2 (4K_2^2 \lambda)^{-1}.$$

For the proof we use Theorem 1 and arguments analogous to ⁽²⁾ (p. 209).

§ 5. Remark. In ⁽⁷⁾ we obtained the following result, supplementing Theorem 1.

Theorem 5. The estimate

$$\sigma_1(\lambda) \leq K_{19}\sqrt{\lambda}, \quad 0 < \lambda < 0.5, \quad (5.1)$$

is valid, where $\sigma_1(\lambda) = \sup_{(f)} S(f)$, the supremum being taken over all meromorphic functions of lower order λ having at least two deficient values.

With the aid of the method of ⁽²⁾, from this theorem we obtain the following result.

Theorem 6. If a meromorphic function $f(z)$ of lower order λ ($\lambda < 0.5$) has at least two deficient values, then

$$\sum_{(a)} \delta(a) \leq K_{20}\lambda^{3/2}. \quad (5.2)$$

This theorem strengthens the result of Edrei and Fuchs ⁽⁸⁾. We have not been able to establish the sharpness of the estimates (5.1) and (5.2). We suppose that in (5.2) the exponent $3/2$ can be replaced by 2.

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