

On the Asymptotic Solution of a System of Linear Differential Equations Containing a Parameter

N. I. Shkil'

1963

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196301.57094>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

N. I. Shkil

ON THE ASYMPTOTIC SOLUTION OF A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS CONTAINING A PARAMETER

(Presented by Academician N. N. Bogolyubov on 10 I 1963)

MATHEMATICS

1. Numerous works have been devoted to the construction of the asymptotic solution of systems of differential equations containing a parameter ⁽¹⁻⁷⁾. Fundamental results were obtained by N. N. Bogolyubov ⁽¹⁾.

Consider, in n -dimensional space, the system of differential equations*

$$\frac{dx}{dt} = A(\tau, \varepsilon)x + \varepsilon B(\tau, \varepsilon)e^{i\theta}, \quad (1)$$

where $A(\tau, \varepsilon)$ is a real matrix of order n ; $B(\tau, \varepsilon)$ is an n -dimensional vector, admitting formal expansions

$$A(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s A^{(s)}(\tau), \quad B(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s B^{(s)}(\tau), \quad 0 \leq \tau = \varepsilon t \leq L; \quad (2)$$

ε is a small real parameter.

The asymptotic solution of system (1) is determined by the behavior of the roots of the equation

$$\det \|A^{(0)}(\tau) - \lambda E\| = 0, \quad (3)$$

where E is the identity matrix.

In ⁽²⁾ a solution was obtained for the case of simple roots on the segment $[0, L]$ of equation (3). The case of multiple roots is partially considered in ^(3,4,7). The asymptotic splitting of system (1) is given in ^(5,6) for arbitrary behavior of the roots of equation (3).

In the present article we consider the case when several multiple elementary divisors correspond to a multiple root.

2. Denote by $\lambda_1(\tau), \lambda_2(\tau), \dots, \lambda_p(\tau)$ the roots of equation (3), having respectively the constant multiplicities $k_1, k_2, \dots, k_p$ ($k_1 + k_2 + \dots + k_p = n$), and suppose that to the root $\lambda_j(\tau)$ ($j = 1, \dots, p$) there correspond r_j elementary divisors

$$(\lambda - \lambda_j(\tau))^{s_{j1}}, \dots, (\lambda - \lambda_j(\tau))^{s_{jr_j}}$$

($s_{j1} + \dots + s_{jr_j} = k_j$; $j = 1, \dots, p$). Then one can indicate a nonsingular matrix $T(\tau)$ such that

$$T^{-1}(\tau)A^{(0)}(\tau)T(\tau) = W(\tau) = \left\| \begin{array}{ccc} W_{k_1}(\tau) & & 0 \\ & \ddots & \\ 0 & & W_{k_p}(\tau) \end{array} \right\|, \quad (4)$$

* Many differential equations are reducible to a system of the form (1), in particular differential equations with a small parameter at the highest derivatives.

where

$$W_{k_j}(\tau) = \left\| \begin{array}{ccc} W_{s_{j1}}(\tau) & & 0 \\ & \ddots & \\ & & W_{s_{jr_j}}(\tau) \end{array} \right\|, \quad W_{s_{jn_j}}(\tau) = \left\| \begin{array}{cccc} \lambda_j(\tau) & 1 & 0 & \dots & 0 \\ 0 & \lambda_j(\tau) & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_j(\tau) & \dots & 0 \end{array} \right\|_{s_{jn_j}}, \quad (5)$$

$$n_j = 1, \dots, r_j; \quad j = 1, \dots, p;$$

$T^{-1}(\tau)$ is the matrix inverse to the matrix $T(\tau)$.

Suppose that the function $i\omega(\tau)$ ($\omega(\tau) = d\theta/dt$, $i = \sqrt{-1}$) at isolated points of the segment $[0, L]$ becomes equal to one of the roots of equation (3), for example $\lambda_1(\tau)$, but $i\omega(\tau) \neq \lambda_k(\tau)$ ($k = 2, \dots, p$) for any $\tau \in [0, L]$. This case is customarily called "resonant."

Theorem 1. *If $A(\tau, \varepsilon)$, $B(\tau, \varepsilon)$, $\omega(\tau)$ have derivatives of all orders with respect to τ , and for all $\tau \in [0, L]$ the ν_{jn_j} -component of the vector*

$$C(\tau) = T^{-1}(\tau) \left[A^{(1)}(\tau)T_{\nu_{jn_j-1}+1}(\tau) - \frac{dT_{\nu_{jn_j-1}+1}(\tau)}{d\tau} \right], \quad (6)$$

where $T_{\nu_{jn_j-1}+1}(\tau)$ is the column of the matrix $T(\tau)$ numbered $\nu_{jn_j-1}+1$ ($\nu_{jn_j} = k_1 + \dots + k_{j-1} + s_{j1} + \dots + s_{jn_j}$; $n_j = 1, \dots, r_j$; $j = 1, \dots, p$), is not equal to zero, then the asymptotic solution of system (1) in the "resonant" case can be represented in the form

$$x = \sum_{n_1=1}^{r_1} [U_{1n_1}(\tau, \mu_{1n_1})h_{1n_1} + P_{n_1}(\tau, \mu_{1n_1})] e^{i\theta} + \sum_{k=2}^p \sum_{n_k=1}^{r_k} U_{kn_k}(\tau, \mu_{kn_k})h_{kn_k}; \quad (7)$$

$$\frac{dh_{1n_1}}{dt} = [\lambda_{1n_1}(\tau, \mu_{1n_1}) - i\omega(\tau)]h_{1n_1} + z_{n_1}(\tau, \mu_{1n_1}), \quad n_1 = 1, \dots, r_1; \quad (8)$$

$$\frac{dh_{kn_k}}{dt} = \lambda_{kn_k}(\tau, \mu_{kn_k})h_{kn_k}, \quad n_k = 1, \dots, r_k; \quad k = 2, \dots, p, \quad (9)$$

where $U_{jn_j}(\tau, \mu_{jn_j})$, $P_{n_1}(\tau, \mu_{1n_1})$ are n -dimensional vectors; h_{jn_j} , $z_{n_1}(\tau, \mu_{1n_1})$ are scalar functions ($n_j = 1, \dots, r_j$; $j = 1, \dots, p$), admitting formal expansions

$$U_{jn_j}(\tau, \mu_{jn_j}) = \sum_{s=0}^{\infty} \mu_{jn_j}^s U_{jn_j}^{(s)}(\tau), \quad \lambda_{jn_j}(\tau, \mu_{jn_j}) = \lambda_j(\tau) + \sum_{s=1}^{\infty} \mu_{jn_j}^s \lambda_{jn_j}^{(s)}(\tau); \quad (10)$$

$$P_{n_1}(\tau, \mu_{1n_1}) = \sum_{s=0}^{\infty} \mu_{1n_1}^s P_{n_1}^{(s)}(\tau), \quad z_{n_1}(\tau, \mu_{1n_1}) = \sum_{s=0}^{\infty} \mu_{1n_1}^s z_{n_1}^{(s)}(\tau); \quad (11)$$

$$\mu_{jn_j} = {}^s j n_j \sqrt{\varepsilon}, \quad n_j = 1, \dots, r_j; \quad j = 1, \dots, p. \quad (12)$$

Substituting the vector x , determined by relations (7)–(9), into system (1) and, in the identity obtained, equating the coefficients of the functions h_{jn_j} and the free terms, we have

$$(A - \lambda_{jn_j} E)U_{jn_j} = \varepsilon U'_{jn_j}, \quad n_j = 1, \dots, r_j; \quad j = 1, \dots, p^*, \quad (13)$$

$$(A - i\omega E)P_{n_1} = U_{1n_1} z_{n_1} + \varepsilon (P'_{n_1} - B), \quad n_1 = 1, \dots, r_1. \quad (14)$$

* Here and below we omit the arguments of the quantities.

To determine the coefficients of the series (10), we shall use relation (13). Equating in it the coefficient of $\mu_{jn_j}^0$, we have

$$(A^{(0)} - \lambda_j E)U_{jn_j}^{(0)} = 0, \quad n_j = 1, \dots, r_j; \quad j = 1, \dots, p. \quad (15)$$

Introducing the vector

$$Q_{jn_j}^{(s)} = T^{-1}U_{jn_j}^{(s)}, \quad s = 0, 1, \dots, \quad (16)$$

equation (15) may be put in the form

$$(W - \lambda_j E)Q_{jn_j}^{(0)} = 0, \quad (17)$$

or, according to (4),

$$(W_{k_r} - \lambda_j E)Q_{jn_j k_r}^{(0)} = 0, \quad n_j = 1, \dots, r_j; \quad r, j = 1, \dots, p, \quad (18)$$

where $Q_{jn_j k_r}^{(0)}$ is the vector with components

$$Q_{jn_j k_r}^{(0)} = (\{q_{jn_j}^{(0)}\}_{l_{r-1}+1}, \{q_{jn_j}^{(0)}\}_{l_{r-1}+2}, \dots, \{q_{jn_j}^{(0)}\}_{l_r}), \quad (19)$$

$$l_r = k_1 + \dots + k_r; \quad r = 1, \dots, p.$$

From (18) we find

$$Q_{jn_j k_r}^{(0)} \equiv 0, \quad j \neq r; \quad j, r = 1, \dots, p; \quad n_j = 1, \dots, r_j. \quad (20)$$

For $r = j$, equation (18) is representable in the form

$$(W_{s_j i_j} - \lambda_j E)Q_{jn_j s_j i_j}^{(0)} = 0, \quad (21)$$

where

$$Q_{jn_j s_j i_j}^{(0)} = (\{q_{jn_j}^{(0)}\}_{v_{i_j j-1}+1}, \{q_{jn_j}^{(0)}\}_{v_{i_j j-1}+2}, \dots, \{q_{jn_j}^{(0)}\}_{v_{i_j j}}), \quad (22)$$

\end{equation}

$$v_{i_j j} = k_1 + \dots + k_{j-1} + s_{j1} + \dots + s_{ji_j};$$

$$n_j, i_j = 1, \dots, r_j; \quad j = 1, \dots, p.$$

Taking (5) into account, from (21) we find

$$\{q_{jn_j}^{(0)}\}_{v_{i_j j-1}+2} = \dots = \{q_{jn_j}^{(0)}\}_{v_{i_j j}} = 0, \quad n_j, i_j = 1, \dots, r_j; \quad j = 1, \dots, p. \quad (23)$$

The component $\{q_{jn_j}^{(0)}\}_{v_{i_j j-1}+1}$ is arbitrary. Put

$$\{q_{jn_j}^{(0)}\}_{v_{i_j j-1}+1} = \begin{cases} 1, & i_j = n_j, \\ 0, & i_j \neq n_j; \quad i_j, n_j = 1, \dots, r_j; \quad j = 1, \dots, p. \end{cases} \quad (24)$$

To determine the vector $Q_{jn_j}^{(1)}$ and the function $\lambda_{jn_j}^{(1)}$, compare in (13) the coefficients of $\mu_{jn_j}^1$. Then, repeating the preceding arguments, we find:

$$Q_{jn_j k_r}^{(1)} \equiv 0, \quad Q_{jn_j s_j i_j}^{(1)} \equiv 0, \quad r \neq j; \quad i_j \neq n_j; \quad i_j, n_j = 1, \dots, r_j; \quad (25)$$

$$r, j = 1, \dots, p;$$

$$\{q_{jn_j}^{(1)}\}_{v_{jn_j-1}+2} = \lambda_{jn_j}^{(1)}, \quad \{q_{jn_j}^{(1)}\}_{v_{jn_j-1}+3} = \dots = \{q_{jn_j}^{(1)}\}_{v_{jn_j}} = 0. \quad (26)$$

The component $\{q_{jn_j}^{(1)}\}_{v_{jn_j-1}+1}$ is arbitrary. We put it equal to zero. Equating now in relation (13) the coefficients successively of $\mu_{jn_j}^2, \dots, \mu_{jn_j}^{s_{jn_j}-1}$ and applying the method of induction, we have

$$\{q_{jn_j}^{(s_{jn_j}-1)}\}_{v_{jn_j}} = (\lambda_{jn_j}^{(1)})^{s_{jn_j}-1}, \quad \{q_{jn_j}^{(\eta_{jn_j})}\}_{v_{jn_j}} = 0, \quad 0 \leq \eta_{jn_j} \leq s_{jn_j} - 2. \quad (27)$$

Then, equating in (13) the coefficients of $\mu_{jn_j}^{s_{jn_j}}$ and taking (27) into account, we find

$$\lambda_{jn_j}^{(1)}(\tau) = {}_{s_{jn_j}}\sqrt{\left\{T^{-1}(\tau) \left[A^{(1)}(\tau) T_{\nu_{jn_j-1}+1}(\tau) - \frac{dT_{\nu_{jn_j-1}+1}(\tau)}{dt} \right] \right\}}_{\nu_{jn_j}}, \quad (28)$$

$$n_j = 1, \dots, r_j; \quad j = 1, \dots, p.$$

All subsequent coefficients of the series (10) are determined in the same way.

The vector $P_{n_1}(\tau, \mu_{1n_1})$ and the function $z_{n_1}(\tau, \mu_{1n_1})$ ($n_1 = 1, \dots, r_1$) are determined from relation (14).

Applying the method set forth in ⁽¹⁾, one can prove the following theorem.

Theorem 2. *If the conditions of Theorem 1 are satisfied and*

$$x|_{t=0} = x_m|_{t=0}; \quad \operatorname{Re} \left(\sum_{s=0}^{s_{jn_j}-1} \mu_{jn_j}^s \lambda_{jn_j}^{(s)} \right) \leq 0, \quad n_j = 1, \dots, r_j; \quad j = 1, \dots, p, \quad (29)$$

where x_m is the vector defined by relations (7)–(12); if the series are cut off after the m -th terms, then for any $L > 0$ and $0 < \mu_{jn_j} \leq \bar{\mu}_{jn_j}$ one can indicate constants C_{jn_j} , independent of the parameter μ_{jn_j} , such that

$$\|x - x_m\| \leq \sum_{j=1}^p \sum_{n_j=1}^{r_j} \mu_{jn_j}^{m+2-2s_{jn_j}} C_{jn_j}. \quad (30)$$

Relation (30) proves the asymptotic character of the solution x_m .

Kyiv State Pedagogical Institute
named after A. M. Gorky

Received
3 I 1963

References

1. N. N. Bogolyubov, *On certain statistical methods in mathematical physics*, 1945.
2. S. F. Feshchenko, *Scientific Notes of the Kyiv Pedagogical Institute*, **9**, No. 4 (1949).
3. Ya. D. Tamarkin, *On certain general problems of the theory of ordinary linear differential equations and on the expansion of arbitrary functions in series*, 1917.
4. Yu. A. Mitropolsky, *Scientific Notes of Kyiv University*, **26**, No. 2, 53 (1957).
5. S. F. Feshchenko, *Reports of the Academy of Sciences of the Ukrainian SSR*, **1**, 11 (1949).

6. Yu. L. Daletskii, S. G. Krein, *Ukrainian Mathematical Journal*, **2**, No. 4, 71 (1950).
7. M. I. Shkil, *Reports of the Academy of Sciences of the Ukrainian SSR*, **9**, 1138 (1962).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.