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Abstract

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MATHEMATICS

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REPRESENTATIONS OF LINEAR CONTINUOUS OPERATORS BY ABSTRACT FUNCTIONS AND EMBEDDING THEOREMS

(Presented by Academician S. L. Sobolev on 6 IV 1963)

1. Let Ω be an L -measurable (bounded or unbounded) set in n -dimensional Euclidean space R_n . Let $\Xi(\Omega)$ be some additive family* of L -measurable subsets of Ω . Denote by $A(\Omega)$ the B -space of numerical functions of the point $y \in \Omega$ satisfying two conditions: a) the totality Λ of all finite linear combinations of characteristic functions of sets $\chi_E(y)$, $E \in \Xi(\Omega)$, is everywhere dense in $A(\Omega)$, i.e. $A(\Omega) = \overline{\Lambda}$; b) the general form of linear functionals over the space $A(\Omega)$ is

$$l(u) = \int_{\Omega} u(y)v(y) dy, \quad u \in A(\Omega), \quad v \in B(\Omega) = (A(\Omega))^*.$$

Examples of such spaces may be: 1) $L_p(\Omega)$, where $\Xi(\Omega)$ is the totality $\Sigma(\Omega)$ of all L -measurable subsets of finite measure from Ω ; 2) $L_{q,\beta}(\Omega)$, Ω a bounded domain, $\Xi(\Omega)$ the totality $\overset{\circ}{\Sigma}(\Omega)$ of all L -measurable subsets lying strictly inside Ω ; 3) $E_N(\Omega)$ ((¹), p. 98), Ω a closed set, $\Xi(\Omega)$ coincides with $\Sigma(\Omega)$.

Definition 1. Let $B(\Omega)$ be the space conjugate to $A(\Omega)$. Denote by $B(X, \Omega)$ the totality of all additive set functions φ , defined on $\Xi(\Omega)$ with values in the B -space X , with finite norm

$$\|\varphi\|_{B(X, \Omega)} = \sup_{\omega \in \Lambda} \frac{\left\| \int_{\Omega} \omega(y) d\varphi \right\|_X}{\|\omega\|_{A(\Omega)}}, \quad (1)$$

where for any $\omega(y) = \sum a_i \chi_{E_i}(y)$ from Λ , by definition,

$$\int_{\Omega} \omega(y) d\varphi = \sum a_i \varphi(E_i).$$

When $A(\Omega) = L_{p'}(\Omega)$, $1 < p' < \infty$, Definition 1 coincides with the definition of the space $\Phi_p(X, \Omega)$ given by S. L. Sobolev ⁽²⁾.

Let $\omega \in \Lambda$, $\varphi \in B(X, \Omega)$; then, by virtue of (1),

$$\left\| \int_{\Omega} \omega(y) d\varphi \right\|_X \leq \|\omega\|_{A(\Omega)} \|\varphi\|_{B(X, \Omega)}. \quad (2)$$

Inequality (2) and property a) make it possible, by means of a limiting passage, to construct the integral $\int_{\Omega} g(y) d\varphi$ for an arbitrary function $g(y)$ from $A(\Omega)$.

* By an additive family is meant a collection of subsets of the given set that contains the union and intersection of any two of its elements.

Theorem 1. The space $B(X, \Omega)$ is isometric and isomorphic to the space $(A(\Omega) \rightarrow X)^*$. The isometric isomorphism has the form

$$T(g) = \int_{\Omega} g(y) d\varphi, \quad g \in A(\Omega), \quad \varphi \in B(X, \Omega). \quad (3)$$

Proof. Let $\varphi \in B(X, \Omega)$. Then the integral in (3) exists and defines a linear continuous operator T . The linearity of the operator is obvious; the equality $\|T\| = \|\varphi\|$ follows from (1) and condition a).

Now let $Q \in (A(\Omega) \rightarrow X)^*$. Consider the additive set function $\varphi(E) = Q(\chi_E(y))$, $E \in \Xi(\Omega)$. Since

$$\|\varphi\| = \sup_{\omega \in \Lambda} \frac{\|\sum \alpha_i \varphi(E_i)\|_X}{\|\sum \alpha_i \chi_{E_i}(y)\|_{A(\Omega)}} = \sup_{\omega \in \Lambda} \frac{\|\sum \alpha_i Q(\chi_{E_i})\|_X}{\|\omega\|_{A(\Omega)}} = \|Q\|,$$

it follows that $\varphi \in B(X, \Omega)$ and, by means of (3), defines some operator T , and for any function ω from Λ ,

$$T(\omega) = T\left(\sum \alpha_i \chi_{E_i}(y)\right) = \sum \alpha_i \varphi(E_i) = \sum \alpha_i Q(\chi_{E_i}) = Q(\omega).$$

Since $\overline{\Lambda} = A(\Omega)$, we have $T = Q$.

Theorem 2. The space $(X \rightarrow B(\Omega))$ is isometric and isomorphic to the space $B(X^*, \Omega)$. The isometric isomorphism is determined by the equality

$$\int_E \mathcal{J}x dy = \varphi(E)x, \quad x \in X, \quad \varphi \in B(X^*, \Omega), \quad E \in \Xi(\Omega). \quad (4)$$

Proof. The equality

$$(Tg, x) = (\mathcal{J}x, g),$$

where $\mathcal{T} \in Y_1 = (X \rightarrow B(\Omega))$, $T \in Y_2 = (A(\Omega) \rightarrow X^*)$, $x \in X$, $g \in A(\Omega)$, defines an isometric isomorphism between the spaces Y_1 and Y_2 . Hence, and from Theorem 1, it follows that between the spaces Y_1 and $B(X^*, \Omega)$ there exists an isometric isomorphism determined by the equality

$$\left(\int_{\Omega} g(y) d\varphi, x \right) = (\mathcal{T}x, g). \quad (5)$$

Putting in (5) $g(y) = \chi_E(y)$, $E \in \Xi(\Omega)$, and using property b), we obtain (4). We note that from (4) it is not difficult to obtain (5), if one uses properties a) and b).

The theorem proved makes it possible to describe the space $B(X, \Omega)$. Denote by $(X^* \rightarrow B(\Omega), \text{ weakly cont.})$ the totality of all operators from Y_1 that take weakly convergent generalized sequences** of functionals from $B(\Omega)$.

Theorem 3. Between the spaces $B(X, \Omega)$ and $(X^* \rightarrow B(\Omega), \text{ weakly cont.})$ there exists an isometric isomorphism, determined by the equality

$$x^* \varphi(E) = \int_E \mathcal{T}x^* dy, \quad x^* \in X^*, \quad \varphi \in B(X, \Omega), \quad E \in \Xi(\Omega). \quad (6)$$

Proof. Introduce into consideration the isometric operator $\varkappa$ ((3), p. 78), defining it by the equality

$$(\varkappa x, x^*) = (x^*, x), \quad x^* \in X^*, \quad x \in X.$$

The equality

$$\varphi^{**}(E) = \varkappa \varphi(E), \quad E \in \Xi(\Omega), \quad (7)$$

establishes an isometric isomorphism between $B(X, \Omega)$ and a certain subspace $\widetilde{B}(X^{**}, \Omega)$ of the space $B(X^{**}, \Omega)$. By Theorem 2, $\widetilde{B}(X^{**}, \Omega)$ is isometric and isomorphic to a certain subspace $\widetilde{W}$ of the space—

* Everywhere in this article, $(Y \rightarrow Z)$ means the space of all linear continuous operators acting from Y to Z .

** For the definition of generalized sequence, see (3), p. 38.

spaces $(X^* \rightarrow B(\Omega))$, and the corresponding elements $\varphi^{**} \in \widetilde{B}(X^{**}, \Omega)$, $\mathcal{T} \in \widetilde{W}$, are connected by the relation

$$\varphi^{**}(E)x^* = \int_E \mathcal{T}x^* dy, \quad E \in \Xi(\Omega). \quad (8)$$

Let $x_\alpha^* \in X^*$, $x_0^* \in X^*$, and for every $x \in X$ $\lim_\alpha x_\alpha^*(x) = x_0^*(x)$.^{*} Then, for every $E \in \Xi(\Omega)$,

$$\lim_\alpha (\mathcal{T} x_\alpha^*, \chi_E) = \lim_\alpha \varphi^{**}(E) x_\alpha^* = \lim_\alpha x_\alpha^* \varphi(E) = x_0^* \varphi(E) = \varphi^{**}(E) x_0^* = (\mathcal{T} x_0^*, \chi_E).$$

Hence, and from a), it follows that $\widetilde{W} = (X^* \rightarrow B(\Omega))$, weakly continuous). Equality (6) follows from (7) and (8).

2. Let Ω be a certain domain in R_n . We shall say that the space $A(\Omega)$ satisfies condition c) if the set $C_0^\infty(\Omega)$ of infinitely differentiable functions finite in Ω is everywhere dense in $A(\Omega)$.

It is clear that the spaces mentioned above, $L_p(\Omega)$, $L_{q,\beta}(\Omega)$, $E_N(\Omega)$, are among such spaces.

Definition 2. Let $A(\Omega)$ satisfy conditions a), b), c). A function $\varphi^{(\gamma)} \in B(X, \Omega)$, $(\gamma) = (\gamma_1, \dots, \gamma_n)$, $|\gamma| = \gamma_1 + \dots + \gamma_n$, is called the **generalized derivative** of the function $\varphi \in B(X, \Omega)$ if, for every function $g \in C_0^\infty(\Omega)$, the equality

$$\int_\Omega \frac{\partial^{(\gamma)} g(y)}{\partial y_1^{\gamma_1} \dots \partial y_n^{\gamma_n}} d\varphi = (-1)^{|\gamma|} \int_\Omega g(y) d\varphi^{(\gamma)}. \quad (9)$$

holds.

From Theorem 1 and properties c) and a) it follows that the generalized derivative is determined uniquely.

Definition 3. Denote by $B^l(X, \Omega)$ the set of all functions $\varphi \in B(X, \Omega)$ having all possible generalized derivatives of order l from $B(\Omega)$. Define the norm in $B^l(X, \Omega)$ by the equality

$$\|\varphi\|_{B^l(X, \Omega)} = \|\varphi\|_{B(X, \Omega)} + \sum_{|\gamma|=l} \|\varphi^{(\gamma)}\|_{B(X, \Omega)}. \quad (10)$$

Let us note that if $A(\Omega) = L_{p'}(\Omega)$, then $B^l(R_1, \Omega) = W_p^l(\Omega)$ and $B^l(X, \Omega) = \Phi_p^l(X, \Omega)$ (^{2,4}). If $A(\Omega) = L_{p', -\alpha}(\Omega)$, then $B^l(R_1, \Omega) = W_{p, \alpha}^l(\Omega)$, and it is natural to denote $B^l(X, \Omega)$ by $\Phi_{p, \alpha}^l(X, \Omega)$. Finally, if $A(\Omega) = E_N(\Omega)$, then $B^l(R_1, \Omega) = F_M^l(\Omega)$; the corresponding space $B^l(X, \Omega)$ shall be denoted by $F_M^l(X, \Omega)$ (for definitions of the spaces $W_p^l(\Omega)$, $W_{p, \alpha}^l(\Omega)$, see, for example, the survey article (⁵), and of the spaces $F_M^l(\Omega)$, in (⁶)).

Using Theorem 2, one can prove the following theorem:

Theorem 4. The space $(X \rightarrow B^l(\Omega))$ is linearly isomorphic to the space $B^l(X^*, \Omega)$. The isomorphism has the form

$$\varphi(E)x = \int_E T x \, dy, \quad \varphi \in B^l(X^*, \Omega), \quad T \in (X \rightarrow B^l(\Omega)), \quad E \in \Xi(\Omega), \quad x \in X,$$

and moreover

$$\|T\| \leq \|\varphi\|_{B^l(X, \Omega)} \leq [1 + M(n, l)] \|T\|, \quad (11)$$

where $M(n, l)$ is the number of terms of the sum in (11).

* The limit is understood in the sense of Moore and Smith (see, for example, (3), p. 38).

A consequence of Theorems 3 and 4 is

Theorem 5. The space $B^l(X, \Omega)$ is linearly isomorphic to the space $(X^* \rightarrow B^l(\Omega), (*))$ of operators $\mathcal{T}$ from $(X^* \rightarrow B^l(\Omega))$ that take every weakly convergent generalized sequence of functionals x_α^* from X^* into weakly convergent generalized sequences $\{\mathcal{T}x_\alpha^*\}$, $\{D^{(\gamma)}\mathcal{T}x_\alpha^*\}$, $\gamma \models l$, of functionals of the space $B(\Omega)$. The isomorphism is defined by the equality

$$x^*\varphi(E) = \int_E \mathcal{T}x^* \, dy, \quad x^* \in X^*, \quad E \in \mathfrak{E}(\Omega), \quad \varphi \in B^l(X, \Omega),$$

and, moreover, inequality (12) is satisfied.

Remark 1. If $B^l(\Omega) = F_M^l(\Omega)$, then $(X^* \rightarrow F_M^l(\Omega), (*))$ is the space of all linear continuous operators that take every weakly convergent generalized sequence of functionals from X^* into an (o) -weakly convergent generalized sequence of elements of $F_M^l(\Omega)$ (the (o) -weak convergence of a generalized sequence is defined by analogy with the (o) -weak convergence of ordinary sequences from $F_M^l(\Omega)$, introduced in (6)).

Remark 2. If in $B^l(\Omega)$ one can introduce an equivalent norm so that it thereby becomes a reflexive space (this can be done, for example, if $B^l(R) = W_p^l(\Omega)$ or $B^l(\Omega) = W_{p, \alpha}^l(\Omega)$), then $(X^* \rightarrow B^l(\Omega), (*))$ is the collection of all linear continuous operators that take every weakly convergent generalized sequence of functionals from X^* into a weakly convergent generalized sequence of elements of $B^l(\Omega)$.

Theorem 6. 1) Let $F_{M_1}^{l_1}(\Omega_{n_1}) \rightarrow F_{M_2}^{l_2}(\Omega_{n_2})$, $n_1 \geq n_2$, and let V be the embedding operator. Then $F_{M_1}^{l_1}(X, \Omega_{n_1}) \rightarrow F_{M_2}^{l_2}(X, \Omega_{n_2})$. 2) Let $W_{p_1, \alpha_1}^{l_1}(\Omega_{n_1}) \rightarrow W_{p_2, \alpha_2}^{l_2}(\Omega_{n_2})$, and let V be a linear continuous operator effecting an embedding (if $n_1 \geq n_2$) or an extension (if $n_1 < n_2$). Then $\Phi_{p_1, \alpha_1}^{l_1}(X, \Omega_{n_1}) \rightarrow \Phi_{p_2, \alpha_2}^{l_2}(X, \Omega_{n_2})$.

Proof. 1) Consider the operator $V\mathcal{T}$, where $\mathcal{T}$ is the operator from $(X^* \rightarrow F_{M_1}^{l_1}(\Omega_{n_1}), (*))$ corresponding to the function $\varphi \in F_{M_1}^{l_1}(X, \Omega_{n_1})$ by Theorem 5. From Remark 1 and from the fact that V takes every (o) -weakly convergent generalized sequence from $F_{M_1}^{l_1}(\Omega_{n_1})$ into an analogous sequence from $F_{M_2}^{l_2}(\Omega_{n_2})$ (this follows from the results of ⁽⁶⁾), it follows that $V\mathcal{T} \in (X^* \rightarrow F_{M_2}^{l_2}(\Omega_{n_2}), (*))$. The function $\varphi \in F_{M_2}^{l_2}(X, \Omega_{n_2})$ corresponding to the operator $V\mathcal{T}$ is the trace of the function $\varphi \in F_{M_1}^{l_1}(\varphi, \Omega_{n_1})$. Moreover,

$$\|\varphi\|_{F_{M_2}^{l_2}(X, \Omega_{n_2})} \leq [1 + M(n_2, l_2)] \|V\mathcal{T}\| \leq \|V\| [1 + M(n_2, l_2)] \|\varphi\|_{F_{M_1}^{l_1}(X, \Omega_{n_1})}.$$

- 2) The second assertion of the theorem is proved by an analogous scheme, using Remark 2 and the fact that V takes any weakly convergent generalized sequence of elements from $W_{p_1, \alpha_1}^{l_1}(\Omega_{n_1})$ into an analogous sequence of elements from $W_{p_2, \alpha_2}^{l_2}(\Omega_{n_2})$.

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