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Abstract

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MATHEMATICS

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APPLICATION OF THE METHOD OF GRIDS TO A HYPERBOLIC SYSTEM OF QUASILIN- EAR EQUATIONS IN THE PLANE

(Presented by Academician S. L. Sobolev on 13 VIII 1962)

In the strip $(0 \leq t \leq t^0, 0 \leq x \leq x^0, -\infty < u_1, \dots, u_n < \infty)$ let us consider the system of equations

$$\begin{aligned}\frac{\partial u_1(t, x)}{\partial t} &= \lambda_1(t, x, u) \frac{\partial u_1(t, x)}{\partial x} + f_1(t, x, u), \\ \frac{\partial u_2(t, x)}{\partial t} &= \lambda_2(t, x, u) \frac{\partial u_2(t, x)}{\partial x} + f_2(t, x, u),\end{aligned}\quad (1)$$

where $u_1, u_2, u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, f_1, f_2 are column matrices; λ_1, λ_2 are diagonal matrices, with the diagonal elements of λ_1 nonnegative, and positive for $x = x^0$, while the diagonal elements of λ_2 are nonpositive, and negative for $x = 0$, with initial conditions

$$u(0, x) = \varphi(x) \quad (2)$$

and boundary conditions

$$\begin{aligned}u_1(t, x^0) &= \alpha_1(t, u_2(t, x^0)) + \int_0^t \beta_1(t, \tau, u(\tau, x^0)) d\tau, \\ u_2(t, 0) &= \alpha_2(t, u_1(t, 0)) + \int_0^t \beta_2(t, \tau, u(\tau, 0)) d\tau.\end{aligned}\quad (3)$$

Analogous systems of linear equations were studied in papers ⁽¹⁻³⁾. In studying classical and generalized solutions we shall need compatibility conditions:

$$\varphi_1(x^0) = \alpha_1(0, \varphi_2(x^0)),$$

$$\varphi_2(0) = \alpha_2(0, \varphi_1(0)); \quad (4)$$

$$\lambda_1(0, x^0, \varphi(x^0))\varphi_1'(x^0) + f_1(0, x^0, \varphi(x^0)) = \frac{\partial \alpha_1}{\partial t} \Big|_{t=0, u_2=\varphi_2(x^0)} + \beta_1(0, 0, \varphi(x^0)), \quad (5)$$

$$\lambda_2(0, 0, \varphi(0))\varphi_2'(0) + f_2(0, 0, \varphi(0)) = \frac{\partial \alpha_2}{\partial t} \Big|_{t=0, u_1=\varphi_1(0)} + \beta_2(0, 0, \varphi(0)).$$

To replace the corresponding boundary-value problem by a difference one, we construct a grid with step h in x and k in t . Then the difference system is written as follows:

$$\begin{aligned} u_1^{i+1j} &= \varkappa \lambda_1^{ij} u_1^{ij+1} + (\varepsilon_1 - \varkappa \lambda_1^{ij}) u_1^{ij} + k f_1^{ij}, \\ u_2^{i+1j} &= -\varkappa \lambda_2^{ij} u_2^{ij+1} + (\varepsilon_2 + \varkappa \lambda_2^{ij}) u_2^{ij} + k f_2^{ij}; \\ u_1^{i+1\nu} &= \alpha_1^{i+1} + k \sum_{p=0}^i \beta_1^{i+1p}, \quad u_2^{i+10} = \alpha_2^{i+1} + k \sum_{p=0}^i \beta_2^{i+1p}, \end{aligned} \quad (6)$$

where $\varkappa = kh^{-1}$; $\varepsilon_1, \varepsilon_2$ are unit matrices.

For a matrix $A = \|a_{pq}\|$ the norm $|A|$ is equal to $\max_q \sum_{p=1}^{p_0} |a_{pq}|$. The matrix

$A(t, x, u)$ satisfies the Lipschitz condition ($A \in \text{Lip}$), if for every $a > 0$ there is a K such that

$$|A(t, x, u) - A(\bar{t}, \bar{x}, \bar{u})| \leq K(|t - \bar{t}| + |x - \bar{x}| + |u - \bar{u}|)$$

for $|u|, |\bar{u}| \leq a$. If in the rectangle $\Pi^* = \Pi_{t^*}(0 \leq t \leq t^*, 0 \leq x \leq x^0)$ the integral relation

$$\iint_{\Pi^*} \left[\left(\frac{\partial v}{\partial t} - \frac{\partial v \lambda}{\partial x} \right) u + v f \right] d\Pi = \int_0^{x^0} v u \Big|_0^{t^*} dx - \int_0^{t^*} v \lambda u \Big|_0^{x^0} dt,$$

holds for $u, \lambda, f \in \text{Lip}$, where v is any continuously differentiable row matrix, then we shall say that u is a generalized solution of system (1).

Theorem 1. If $\lambda, f, \varphi, \alpha, \beta$ have first partial derivatives satisfying the Lipschitz condition, and the compatibility conditions (4)–(5) are fulfilled, then there exists a $t^* > 0$ such that in Π^* there exists a classical solution of the boundary-value problem (1)–(3), and moreover $du/dt, du/dx \in \text{Lip}$.

Theorem 2. If $\lambda, f, \varphi, \alpha, \beta \in \text{Lip}$ and the compatibility conditions (4) are fulfilled, then there exists a $t^* > 0$ such that in Π^* there exists a generalized solution of the boundary-value problem (1)–(3), and moreover $u \in \text{Lip}$.

The proofs of these theorems are based on the following lemmas.

Lemma 1. If $\lambda, f, \varphi, \alpha, \beta \in \text{Lip}$ and the compatibility conditions (4) are fulfilled, then there exist U_1, U_2 and $t^2 > 0$ such that, if for the mesh the relations $\kappa|\lambda^{ij}| \leq 1$, $0 \leq a \leq \kappa|\lambda_1^i|^{-1}$, $a \leq \kappa|\lambda_2^i|$ are fulfilled in Π_{t^1} , then $|u^{ij}| \leq U_1$, $|\Delta_1 u^{ij}| \leq U_2$, $|\Delta_2 u^{ij}| \leq U_2$ in Π_{t^*} , where $t^* = \min\{t^1, t^2\}$, $\Delta_1 u^{ij} = (u^{i+1,j} - u^{ij})k^{-1}$, $\Delta_2 u^{ij} = (u^{i,j+1} - u^{ij})h^{-1}$.

Lemma 2. If $\lambda, f, \varphi, \alpha, \beta$ have first partial derivatives satisfying the Lipschitz condition, and the compatibility conditions (4)–(5) are fulfilled, then there exist U_3 and $t^3 > 0$ such that, if for the mesh the relations $\kappa|\lambda^{ij}| \leq 1$, $0 < a \leq \kappa|\lambda_1^i|^{-1}$, $a \leq \kappa|\lambda_2^i|$ are fulfilled in Π_{t^2} , then $|\Delta_1^2 u^{ij}| \leq U_3$, $|\Delta_1 \Delta_2 u^{ij}| \leq U_3$, $|\Delta_2^2 u^{ij}| \leq U_3$ in Π_{t^*} , where $t^* = \max\{t^2, t^3\}$.

Lemma 3. If $\lambda, f \in \text{Lip}$ and a sequence u^p of solutions of system (6) for meshes whose step tends to zero converges uniformly in Π^* to u , and $|\Delta_1 u^p| \leq U_2$, $|\Delta_2 u^p| \leq U_2$, then u is a generalized solution of system (1).

Theorem 3. If $\lambda, f, \varphi, \alpha, \beta \in \text{Lip}$ and the compatibility conditions (4) are fulfilled, then the generalized solution of the boundary-value problem (1)–(3) is unique.

Theorem 4. If $\lambda, \lambda_\varepsilon, f, f_\varepsilon, \varphi, \varphi_\varepsilon, \alpha, \alpha_\varepsilon, \beta, \beta_\varepsilon \in \text{Lip}$, the compatibility conditions (4) are fulfilled, the generalized solution exists in Π^* , $|\lambda - \lambda_\varepsilon| = O(\varepsilon), \dots, |\beta - \beta_\varepsilon| = O(\varepsilon)$, then, for sufficiently small ε , from the existence of the generalized solution u_ε in Π^* it follows that $|u - u_\varepsilon| = O(\varepsilon)$.

Theorem 5. Let u be a classical solution in Π^* of the boundary-value problem (1)–(3), $du/dt, du/dx \in \text{Lip}$. If $\lambda, f, \varphi, \alpha, \beta \in \text{Lip}$, then for sufficiently small h there is a C such that from $\kappa|\lambda^{ij}| \leq 1$ it follows that

$$|u(t_i, x_j) - u^{ij}| \leq Ch.$$

Theorem 6. Let u be a generalized solution in Π^* of the boundary-value problem (1)–(3). If $\lambda, f, \varphi, \alpha, \beta \in \text{Lip}$, then for sufficiently small h there is a C such that from $\kappa|\lambda^{ij}| \leq 1$ it follows that

$$|u(t_i, x_j) - u^{ij}| \leq Ch^{1/2}.$$

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CITED LITERATURE

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