

# On the Simultaneous Approximation of Almost-Periodic Functions and Their Derivatives

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## Abstract

## Full Text

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MATHEMATICS

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# ON THE SIMULTANEOUS APPROXIMATION OF ALMOST-PERIODIC FUNCTIONS AND THEIR DERIVATIVES

(Presented by Academician V. I. Smirnov on 16 II 1962)

1. Let  $W^{(r)}$  be the class of all functions bounded on  $(-\infty, \infty)$  that have a derivative of order  $r$  bounded on the entire real axis; let  $P^{(r)}$  and  $P_{2\pi}^{(r)}$  be, respectively, the classes of all periodic and all  $2\pi$ -periodic functions from  $W^{(r)}$ . We shall say that a uniformly almost-periodic function  $f(x)$  belongs to the class  $A_s$  if  $N_f(x, x+1) = O(1)$ , where  $N_f(x, x+1)$  is the number of Fourier exponents of the function  $f(x)$  on the interval  $(x, x+1)$ . (The structural characteristic of the class  $A_s$  is given in <sup>(5)</sup>.) Denote by  $W_s^{(r)}$  the intersection of the classes  $W^{(r)}$  and  $A_s$ . Obviously,  $W_s^{(r)} \supset P_{2\pi}^{(r)}$ . Set:

$$C_{\sigma, r}(f) = \inf_{g_\sigma(x)} \max_{0 \leq k \leq r} \frac{\sup_x |f^{(k)}(x) - g_\sigma^{(k)}(x)|}{E_\sigma(f^{(k)})},$$

where  $g_\sigma(x)$  is an entire function of degree  $\leq \sigma$  and

$$E_\sigma(f) = \inf_{g_\sigma(x)} \sup_x |f(x) - g_\sigma(x)|;$$

$$C_{n, r}^*(f) = \inf_{T_n(x)} \max_{0 \leq k \leq r} \frac{\sup_x |f^{(k)}(x) - T_n^{(k)}(x)|}{E_n^*(f^{(k)})},$$

where

$$T_n(x) = \sum_{\nu=0}^n a_\nu \cos \nu x + b_\nu \sin \nu x$$

and

$$E_n^*(f) = \inf_{T_n(x)} \sup_x |f(x) - T_n(x)|.$$

Let

$$C_{\sigma,r}(W^{(r)}) = \sup_{f \in W^{(r)}} C_{\sigma,r}(f);$$

analogously the quantities  $C_{\sigma,r}(W_s^{(r)})$ ,  $C_{\sigma,r}(P^{(r)})$ ,  $C_{n,r}^*(P_{2\pi}^{(r)})$  are defined.

A. F. Timan <sup>(1)</sup> showed that, as  $r \rightarrow \infty$ , uniformly with respect to all  $\sigma > 0$ , the following asymptotic equality holds:

$$C_{\sigma,r}(W^{(r)}) = \frac{4}{\pi^2} \ln(r+1) + O(\ln \ln \ln r). \quad (1)$$

A. L. Garkavi <sup>(2)</sup> obtained the asymptotic formula

$$C_{n,r}^*(P_{2\pi}^{(r)}) = \frac{4}{\pi^2} \ln(p+1) + O(\ln \ln \ln p), \quad (2)$$

where  $p = \min\{n, r\}$ .

It follows from the theorems given below that the result of A. L. Garkavi can be extended to the class  $W_s^{(r)}$  of almost-periodic functions, while estimate (1) for the quantity  $C_{\sigma,r}$  for the class of all uniformly almost-periodic functions belonging to  $W^{(r)}$  cannot be improved, since it cannot be improved even for the class  $P^{(r)}$  contained in it.

2. We formulate the main results of the note.

**Theorem 1.** Whatever the positive real  $\sigma > 0$  and the natural number  $r$ , as  $p \rightarrow \infty$  the asymptotic equality

$$C_{\sigma,r}(W_s^{(r)}) = \frac{4}{\pi^2} \ln(p+1) + O(\ln \ln \ln p), \quad \text{where } p = \min\{\sigma, r\}. \quad (3)$$

holds.

**Theorem 2.** As  $r \rightarrow \infty$ , uniformly with respect to all  $\sigma > 0$ , the asymptotic equality

$$C_{\sigma,r}(P^{(r)}) = \frac{4}{\pi^2} \ln(r+1) + O(\ln \ln \ln r). \quad (4)$$

holds.

3. Let us recall some known facts and give three lemmas on which the proof of Theorem 1 is based.

The integral operator of N. I. Akhiezer–B. M. Levitan <sup>(3)</sup>

$$f_{\sigma,q}(x) = \int_{-\infty}^{\infty} f(x+u) \Psi_{\sigma,\sigma(1+1/q)}(u) du, \quad (5)$$

where

$$\Psi_{\sigma,\sigma(1+1/q)}(u) = \frac{q}{\pi\sigma} \frac{\cos \sigma u - \cos \sigma(1+1/q)u}{u^2}$$

( $q > 0$ ), assigns to every continuous and bounded function  $f(x)$  on  $(-\infty, \infty)$  an entire function  $f_{\sigma,q}(x)$  of degree  $\leq \sigma(1+1/q)$ , and moreover  $f_{\sigma,q}(x) = f(x)$  if  $f(x)$  is a bounded entire function on  $(-\infty, \infty)$  of degree  $\leq \sigma$ .

Let  $L(q)$  be the norm of the operator (5) in the space of all bounded functions on the real axis. A. F. Timan <sup>(4)</sup> established that, uniformly with respect to all  $q > 0$ , the asymptotic equality

$$L(q) = \frac{4}{\pi^2} \ln(q+1) + O(1) \quad (6)$$

holds.

Let the interval  $I_{\sigma,q} = (\sigma, \sigma(1+1/q))$  contain  $n$  ( $n \geq 0$ ) points  $c_i$  ( $c_i < c_{i+1}$ ,  $i = 1, 2, \dots, n$ ); let  $\varepsilon > 0$  be chosen so that the intervals  $(c_i - \varepsilon, c_i + \varepsilon)$  do not intersect and belong to the interval  $I_{\sigma,q}$ . Consider the function, continuous and linear on the intervals  $(c_i - \varepsilon, c_i)$ ,  $(c_i, c_i + \varepsilon)$  ( $i = 1, 2, \dots, n$ ),

$$\begin{aligned} \varphi_{\sigma,q}(\lambda) &= \varphi_{\sigma,q}(\lambda, c_1, c_2, \dots, c_n, \varepsilon) = \\ &= \begin{cases} 1, & |\lambda| \leq \sigma, \\ q+1 - \frac{q}{\sigma}|\lambda|, & \sigma < |\lambda| < \sigma\left(1 + \frac{1}{q}\right), \quad |\lambda| \notin (c_i - \varepsilon, c_i + \varepsilon), \\ 0, & |\lambda| = c_i, \\ 0, & |\lambda| \geq \sigma\left(1 + \frac{1}{q}\right). \end{cases} \end{aligned}$$

Set

$$\begin{aligned} \widetilde{\Psi}_{\sigma,\sigma(1+1/q)}(u) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_{\sigma,q}(\lambda) e^{-iu\lambda} d\lambda, \\ \widetilde{f}_{\sigma,q}(x) &= \int_{-\infty}^{\infty} f(x+u) \widetilde{\Psi}_{\sigma,\sigma(1+1/q)}(u) du; \end{aligned} \quad (5')$$

it is easy to see that  $\tilde{\Psi}_{\sigma, \sigma(1+1/q)}(u) = \Psi_{\sigma, \sigma(1+1/q)}(u)$ ,  $\tilde{f}_{\sigma, q}(x) = f_{\sigma, q}(x)$ , if the interval  $I_{\sigma, q}$  is free of the points  $c_i$ .

Let  $\tilde{L}(q)$  be the norm of the operator (5'), and  $\tilde{\tilde{L}}(q)$  the norm of the operator

$$\int_{-\infty}^{\infty} f(x+u) [\tilde{\Psi}_{\sigma, \sigma(1+1/q)}(u) - \Psi_{\sigma(1-1/q), \sigma}(u)] du$$

in the space of all functions bounded on the real axis.

**Lemma 1.** If  $f(x)$  is continuous and bounded on  $(-\infty, \infty)$ , then  $\tilde{f}_{\sigma, q}(x)$  is an entire function of degree  $\leq \sigma(1+1/q)$ . If  $f(x)$  is an entire function of degree  $\leq \sigma$  bounded on  $(-\infty, \infty)$ , then  $\tilde{f}_{\sigma, q}(x) = f(x)$ .

**Lemma 2.** The inequalities hold

$$\tilde{L}(q) \leq \bar{L}(q) + 2n, \quad (7)$$

$$\tilde{\tilde{L}}(q) \leq 2(n+1). \quad (8)$$

**Lemma 3.** If

$$f(x) \sim \sum_{k=-\infty}^{\infty} A_k e^{i\lambda_k x} \quad (\lambda_0 = 0, \lambda_{-k} = -\lambda_k, \lambda_k < \lambda_{k+1} \text{ for } k \geq 0, \lim_{k \rightarrow \infty} \lambda_k = \infty)$$

is a uniformly almost-periodic function, then

$$\tilde{f}_{\sigma, q}(x) = \sum_{|\lambda_k| < \sigma(1+1/q)} \varphi_{\sigma, q}(\lambda_k) A_k e^{i\lambda_k x}.$$

4. We shall give the proof of Theorem 1. Let  $I_{\sigma}$  be the common part of the intervals  $I_{\sigma, r}$  and  $(\sigma, \sigma+1)$ ; let  $c_1, c_2, \dots, c_n$  ( $0 \leq n \leq N_f(\sigma, \sigma+1)$ ) be the Fourier exponents of the function  $f(x)$  belonging to the interval  $I_{\sigma}$ . We shall show that for every function  $f(x) \in W_s^{(r)}$

$$\max_{0 \leq k \leq r} \frac{\sup_x |f^{(k)}(x) - G_{\sigma}^{(k)}(\tilde{f}_{\sigma, r}, x)|}{E_{\sigma}(f^{(k)})} \leq \frac{4}{\pi^2} \ln(p+1) + O(1), \quad (9)$$

where  $p = \min\{\sigma, r\}$ ;  $G_{\sigma}(\tilde{f}_{\sigma, r}, x)$  is an entire function of degree  $\leq \sigma$ , realizing the best approximation to the function  $\tilde{f}_{\sigma, r}(x)$ .

The equalities

$$\tilde{f}_{\sigma,r}^{(k)}(x) = \int_{-\infty}^{\infty} f^{(k)}(x+u) \tilde{\Psi}_{\sigma,\sigma(1+1/r)}(u) du \quad (k=0,1,\dots,r) \quad (10)$$

are valid.

Denote by  $g_{\sigma}(f^{(k)}, x)$  an entire function of degree  $\leq \sigma$ , realizing the best approximation to the function  $f^{(k)}(x)$ . By Lemma 1,

$$g_{\sigma}(f^{(k)}, x) = \int_{-\infty}^{\infty} g_{\sigma}(f^{(k)}, x+u) \tilde{\Psi}_{\sigma,\sigma(1+1/q)}(u) du \quad (k=0,1,\dots,r) \quad (11)$$

for any  $q > 0$ .

Let  $\sigma \leq r$ . In consequence of (10) and Lemma 3,

$$\tilde{f}_{\sigma,r}^{(k)}(x) = \int_{-\infty}^{\infty} f^{(k)}(x+u) \tilde{\Psi}_{\sigma,\sigma+1}(u) du = S_{\sigma}^{(k)}(f, x), \quad (12)$$

where

$$S_{\sigma}(f, x) = \sum_{|\lambda_k| \leq \sigma} A_k e^{i\lambda_k x},$$

and therefore

$$G_{\sigma}^{(k)}(\tilde{f}_{\sigma,r}, x) = \tilde{f}_{\sigma,r}^{(k)}(x) \quad (k=0,1,\dots,r). \quad (13)$$

Putting  $q = \sigma$  in equality (11), we obtain from (11), (12), and (13)

$$\sup |f^{(k)}(x) - G_{\sigma}^{(k)}(\tilde{f}_{\sigma,r}, x)| \leq \{\tilde{L}(\sigma) + 1\} E_{\sigma}(f^{(k)}) \quad (k=0,1,\dots,r).$$

From the last inequality, by virtue of (7), it follows that for  $\sigma \leq r$

$$\max_{0 \leq k \leq r} \frac{\sup_x |f^{(k)}(x) - G_{\sigma}^{(k)}(\tilde{f}_{\sigma,r}, x)|}{E_{\sigma}(f^{(k)})} \leq L(\sigma) + 2N_f(\sigma, \sigma+1) + 1. \quad (14)$$

Let  $\sigma > r$ . It is easy to see that

$$\begin{aligned} \int_{-\infty}^{\infty} [f(x+u) - g_{\sigma}(f, x+u)] [\tilde{\Psi}_{\sigma,\sigma(1+1/r)}(u) - \Psi_{\sigma(1-1/r),\sigma}(u)] du = \\ = \tilde{f}_{\sigma,r}(x) - \Phi_{\sigma}(x), \end{aligned}$$

where  $\Phi_\sigma(x)$  is some entire function of degree  $\leq \sigma$ ; therefore, from inequality (8) one obtains the estimate

$$E_\sigma(\tilde{f}_{\sigma,r}) \leq 2[1 + N_f(\sigma, \sigma + 1)]E_\sigma(f). \quad (15)$$

For any function  $f(x) \in W^{(r)}$  (see (1, 3)),

$$E_\sigma(f) \leq \frac{\pi}{2\sigma^k} E_\sigma(f^{(k)}) \quad (k = 0, 1, \dots, r). \quad (16)$$

From S. N. Bernstein's inequality (3) and inequalities (15) and (16) it follows that

$$\begin{aligned} \sup_x |\tilde{f}_{\sigma,r}^{(k)}(x) - G_\sigma^{(k)}(\tilde{f}_{\sigma,r}, x)| &\leq \\ &\leq \pi \left(1 + \frac{1}{r}\right)^k [1 + N_f(\sigma, \sigma + 1)] E_\sigma(f^{(k)}) \quad (k = 0, 1, \dots, r). \end{aligned}$$

Taking  $q = r$  in equality (11), we obtain from (10), (11), and (7)

$$\begin{aligned} \sup_{x'} |\tilde{f}^{(k)}(x) - \tilde{f}_{\sigma,r}^{(k)}(x)| &\leq [L(r) + 2N_f(\sigma, \sigma + 1) + 1] E_\sigma(f^{(k)}) \\ &(k = 0, 1, \dots, r). \end{aligned}$$

From the last two inequalities it follows that, for  $\sigma > r$ ,

$$\max_{0 \leq k \leq r} \frac{\sup_x |f^{(k)}(x) - G_\sigma^{(k)}(\tilde{f}_{\sigma,r}, x)|}{E_\sigma(f^{(k)})} \leq L(r) + (2 + \pi e)N_f(\sigma, \sigma + 1) + \pi e + 1. \quad (17)$$

Estimate (9) is a consequence of inequalities (14), (17), and the asymptotic equality (6). From (9) we obtain the inequality

$$C_{\sigma,r}(W_s^{(r)}) \leq \frac{4}{\pi^2} \ln(p + 1) + O(1), \quad \text{where } p = \min\{\sigma, r\},$$

which, by virtue of (2), leads to the asymptotic equality (3).

**5.** We do not present here the proof of Theorem 2, based on the effective construction of a function  $g_\sigma(x) \in P^{(r)}$  for which

$$C_{\sigma,r}(g_\sigma) \geq \frac{4}{\pi^2} \ln(r + 1) + O(\ln \ln \ln r).$$

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## CITED LITERATURE

1. A. F. Timan, *Izv. AN SSSR, Ser. Mat.*, **24**, No. 3 (1960).
2. A. L. Garkavi, *Izv. AN SSSR, Ser. Mat.*, **24**, No. 1 (1960).
3. N. I. Akhiezer, *Lectures on Approximation Theory*, Moscow-Leningrad, 1947.
4. A. F. Timan, *DAN*, **64**, No. 2 (1949).
5. E. A. Bredikhina, *Matem. sbornik*, **50** (92), 3 (1960).

*Note: Figure translations are in progress. See original paper for figures.*

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