

On the Stability of a Plasma Cylinder in the Case of a Nonuniform Current Distribution over the Cross Section

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Abstract

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PHYSICS

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ON THE QUESTION OF THE STABILITY OF A PLASMA CYLINDER IN THE CASE OF A NONUNIFORM DISTRIBUTION OF CURRENT OVER THE CROSS SECTION

(Presented by Academician B. P. Konstantinov, 25 XII 1961)

We shall study the stability of a plasma cylinder of radius a , assuming that the equilibrium distribution of the magnetic field inside the cylinder and outside it in vacuum has the form:

$$\frac{\mathbf{H}}{H_0} = \begin{cases} \mathbf{i}_\varphi g \frac{r}{d} + \mathbf{i}_z h, & \text{for } 0 \leq r \leq d, \\ \mathbf{i}_\varphi \frac{r}{a} \left(g + f \ln \frac{r}{d} \right) + \mathbf{i}_z h \left(1 + l \ln \frac{r}{d} \right), & \text{for } d \leq r \leq a, \\ \mathbf{i}_\varphi \frac{a}{r} + \mathbf{i}_z h \left(1 + l \ln \frac{a}{d} \right), & \text{for } a \leq r < \infty. \end{cases} \quad (1)$$

Here $g = 1 - f \ln \frac{a}{d}$, $\mathbf{i}_r, \mathbf{i}_\varphi, \mathbf{i}_z$ are the unit vectors of the cylindrical coordinate system; d, f, g, h, l, H_0 are constants such that the pressure $p \geq 0$.

We shall assume that the medium inside the cord may be regarded as incompressible, inviscid, and perfectly conducting. We take the dependence of a perturbation on time and on φ, z in the form $e^{i(\omega t + m\varphi + kz)}$. Introducing the displacement $\vec{\xi} = \mathbf{v}/i\omega$ and the perturbation of the total pressure (hydrodynamic plus magnetic) P^* , we write the initial linearized system of equations of magnetohydrodynamics in the form

$$\frac{H_0^2}{a^2} (s^2 - \Omega^2) \vec{\xi} = -4\pi \nabla P^* + \mathbf{i}_r \xi_r \frac{d}{dr} \left(\frac{H_\varphi}{r} \right)^2 - \frac{2isH_0 H_\varphi}{ar} (\mathbf{i}_r \xi_\varphi - \mathbf{i}_\varphi \xi_r); \quad (2)$$

$$\operatorname{div} \vec{\xi} = 0, \quad (3)$$

where

$$s = \frac{a}{H_0} \left(\frac{mH_\varphi}{r} + kH_z \right), \quad \Omega^2 = \frac{4\pi\rho a^2\omega^2}{H_0^2}.$$

We restrict ourselves to the study of long-wavelength perturbations with $m \geq 1$, so that $|ka| \ll 1$. We shall neglect $k^2 r^2$ in comparison with m^2 ; then, as is not difficult to show, the term with ξ_z in equation (3) may be omitted.* Expressing all unknowns through $X = r\xi_r$ and substituting in (2)–(3) the distribution (1) for the region $d \leq r \leq a$, we obtain

$$(s^2 - \Omega^2) \frac{d^2 X}{ds^2} + 2s \frac{dX}{ds} - \left\{ \frac{s^2 - \Omega^2}{(b+f)^2} + \frac{M + Ns}{b+f} \right\} X = 0, \quad (4)$$

where

$$M = \frac{2b^2(lg - f)l}{(b+f)^2}, \quad N = \frac{2f(2b+f)}{m(b+f)^2}, \quad b = \frac{kahl}{m},$$

$$s = s_0 + m(b+f) \ln \frac{r}{d}, \quad s_0 = mg + kah,$$

and it is assumed that $b+f \neq 0$.†

* In the present work the case is not studied in which, in the interval under consideration, there is a value $s^2 = \Omega^2$. Near such a point various small terms discarded in the initial equations of magnetohydrodynamics may be significant.

The boundary conditions for $X(s)$ are found from the matching condition with the solution in the region $r \leq d$, where $X \sim r^m$, and from the condition of equality of the total pressures on the perturbed surface of the column:

$$\left(\frac{d \ln X}{ds} \right)_{\substack{r=d \\ s=s_0}} = \frac{1}{b+f}; \quad (5)$$

$$\left(\frac{d \ln X}{ds} \right)_{\substack{r=a \\ s=s_1}} = \frac{s_1(2-s_1)}{(b+f)(s_1^2 - \Omega^2)}. \quad (6)$$

Let us first consider the stability of a cylinder with a homogeneous longitudinal field ($l = 0$) with respect to perturbations $m = 1$. For $\Omega^2 < 0$, solution (4) has the form

$$X = e^{s/f} \left\{ A_1 + A_2 \int_{s_0/f}^{s/f} \frac{e^{-2y} dy}{y^2 - \Omega^2/f^2} \right\},$$

$$A^i = \text{const.}$$

Fig. 1

Figure 1: Fig. 1

With the aid of (5)–(6) we obtain

$$\begin{aligned}\Omega^2 &= 2s_1(s_1 - 1) = \\ &= 2kah(1 + kah).\end{aligned}\quad (7)$$

The expression for the oscillation frequency does not depend on the parameter f , which characterizes the current distribution over the cross section. Formula (7) coincides with that which holds for $f = 0$ [1].

Fig. 1

We now turn to the analysis of stability for the case of a longitudinal current homogeneous over the cross section ($f = 0$, $l \neq 0$). For the boundary of the stability region ($\Omega = 0$), taking into account that the solutions of (4) are* $\frac{1}{\sqrt{s}} I_{\pm\nu} \left(\frac{s}{b} \right)$, where

$$\nu = \frac{1}{2} \sqrt{1 + \frac{8}{b}},$$

we find

$$\begin{aligned}\left\{ I'_{\nu} \left(\pm \frac{s_1}{b} \right) \pm \left[1 - \frac{(4\nu^2 + 1)b}{4s_1} \right] I_{\nu} \left(\pm \frac{s_1}{b} \right) \right\} \left\{ K'_{\nu} \left(\pm \frac{s_0}{b} \right) + \left(1 + \frac{b}{2s_0} \right) K_{\nu} \left(\pm \frac{s_0}{b} \right) \right\} = \\ = \left\{ K'_{\nu} \left(\pm \frac{s_1}{b} \right) \pm \left[1 - \frac{(4\nu^2 + 1)b}{4s_1} \right] K_{\nu} \left(\pm \frac{s_1}{b} \right) \right\} \times \\ \times \left\{ I'_{\nu} \left(\pm \frac{s_0}{b} \right) \mp \left(1 + \frac{b}{2s_0} \right) I_{\nu} \left(\pm \frac{s_0}{b} \right) \right\}.\end{aligned}\quad (8)$$

Here the prime denotes differentiation with respect to the argument of the Bessel functions, and it is assumed that the point $s = 0$ does not lie inside the interval s_0, s_1 , so that the sign should be chosen from the condition $\pm s_i/b > 0$.

Equation (8) determines a one-parameter family of curves $s_0(b)$, where $s_0 = m + kah$, $b = kahl/m$. As the parameter one may take the quantity $m \ln(a/d) = (s_1 - s_0)/b$. For $a = d$, instability occurs in the band $0 < s_0 < 1$, which agrees with the result following from the formulas of paper [1]. For the case $m \ln(a/d) = 1/4$, the curves $s_0(b)$ are shown in Fig. 1

* For arbitrary f and l , and $\Omega = 0$, $X(s)$ is expressed in terms of a degenerate hypergeometric function.

(we are speaking of those regions where the signs of s_0 and s_1 are the same). The numbers below the abscissa axis give the values of the parameter ν . The region of instability in the figure is shaded.

Serious difficulties arise in the analysis of stability in the case when $s(r)$ in the interval $d \ll r \ll a$ passes through zero. In order that the initial equation (4) have no singularities, it is necessary to take $\Omega^2 < 0$.

For $f = 0$ and $s^2 \ll b^2$, the solutions of (4) are spherical Legendre functions defined in the complex plane s/Ω with cuts from $-\infty$ to -1 and from 1 to ∞ . Putting

$$\mu = \frac{1}{2} \sqrt{1 + \frac{8}{b} - \frac{4\Omega^2}{b^2}},$$

we obtain

$$X = B_1 P_{-1/2+\mu} \left(\frac{s}{\Omega} \right) + B_2 P_{-1/2+\mu} \left(-\frac{s}{\Omega} \right), \quad B_i = \text{const.} \quad (9)$$

Let us first assume that μ is not close to an integer and that the oscillation frequency satisfies the condition $(-\Omega^2) \ll s_{\max}^2 \ll b^2$. Let us find how $X(s)$ changes on passing through the point $s = 0$. If for $s < 0$, $|s/\Omega| \gg 1$, the solution has the form

$$X = C_1 (-s)^{\mu-1/2} + C_2 (-s)^{-\mu-1/2}, \quad C_i = \text{const},$$

then for values $s > 0$, $|s/\Omega| \gg 1$, with the aid of (9) we find

$$X = \frac{s^{\mu-1/2}}{\sin \pi \mu} \left[C_1 - C_2 \left(-\frac{\Omega^2}{4} \right)^{-\mu} \frac{\Gamma(\mu)\Gamma(-\mu+1/2)}{\Gamma(-\mu)\Gamma(\mu+1/2)} \cos \pi \mu \right] - \frac{s^{-\mu-1/2}}{\sin \pi \mu} \left[C_2 - C_1 \left(-\frac{\Omega^2}{4} \right)^{\mu} \frac{\Gamma(-\mu)\Gamma(\mu+1/2)}{\Gamma(\mu)\Gamma(-\mu+1/2)} \cos \pi \mu \right].$$

One of the constants C_1 or C_2 may be taken equal to unity. For sufficiently small $|\Omega^2|$ there is no dependence on the ratio C_2/C_1 , and stability is determined only by the behavior of $X(s)$ near $s = 0$. For imaginary μ , because of the strong oscillation, one can always find solutions corresponding to small negative Ω^2 —a result agreeing with Suydam's theorem ⁽²⁾. This region is shaded in the figure. For real μ , in accordance with theorem ⁽²⁾, the solutions under consideration do not exist in the limiting case $|\Omega^2| \rightarrow 0$.*

With increasing oscillation frequency $|\Omega|$, the situation may change. Suppose, for example, that $|s_0| \ll 1$, $|s_1| \ll 1$, and that Ω^2 is of the order of s_0 and s_1 . Substituting (9) into conditions (5)–(6), we find

$$\Omega^2 = -2s_0. \quad (10)$$

It follows from this that in the region $0 < \nu < 1/2$, stable according to Suydam's criterion ⁽²⁾, unstable oscillations may in fact arise.

If the oscillation frequency is so large that the conditions

$$\left| \frac{b}{s^2 - \Omega^2} \right| \ll 1, \quad \left| \frac{bs}{s^2 - \Omega^2} \right| \ll 1, \quad |b| \ll 1,$$

are satisfied, then the solutions of (4) for $f = 0$ are written as series in powers

$$e^{\pm s/b} \frac{1}{\sqrt{s^2 - \Omega^2}} \left(\frac{s - \Omega}{s + \Omega} \right)^{\pm 1/2 \Omega} \left\{ 1 \mp \frac{b}{2} \int \frac{1 + \Omega^2 \mp 2s}{(s^2 - \Omega^2)} ds + \dots \right\},$$

and with the aid of (5)–(6) we find

$$\begin{aligned} \Omega^2 = & 2s_1(s_1 - 1) - b(s_1 - 1) \\ & + b(s_0 - 1) \frac{s_1^2 - \Omega^2}{s_0^2 - \Omega^2} \left(\frac{s_1 + \Omega}{s_1 - \Omega} \frac{s_0 - \Omega}{s_0 + \Omega} \right)^{1/\Omega} \left(\frac{a}{d} \right)^{-2m} + \dots \end{aligned} \quad (11)$$

* It should be noted that for small oscillation frequencies near the point $s = 0$, the nonideal conductivity of the plasma may be important.

Let $s_0 s_1 < 0$ and $mb \ln(a/d)$ be of order unity. For small b , formula (11) gives a distribution of the regions of stability and instability opposite to that given by Suydam's theorem ². For example, if $s_0 = -s_1$, $b > 0$, $mb \ln(a/d) < 2$, $m \ln(a/d) \gg 1$, then $\Omega^2 < 0$, whereas according to Suydam's criterion ² one should have $\Omega^2 > 0$.

Thus, the picture shown in Fig. 1 is incomplete, since the curve corresponding to this negative value of Ω^2 does not necessarily lie inside the instability region hatched in the figure.

Let us also consider the transition to a filament with a surface current, when $b + f \gg 1$, $a \simeq d$, and $(b + f) \ln(a/d)$ is a finite quantity. We assume that $\ln(-\Omega^2) \ll (b + f)$. Taking the principal terms into account,

$$X = D_1 \left\{ 1 + \frac{2}{2(b + f)} \left[M \ln \left(1 - \frac{s^2}{\Omega^2} \right) + Ns \right] \right\} + D_2 \arctg \frac{is}{\Omega}, \quad D_i = \text{const.}$$

Substituting into conditions (5)–(6), we find

$$\Omega^2 = s_1^2 + s_0^2 - 2gs_0 - m(1 - g^2). \quad (12)$$

In the present case, irrespective of whether $s(r)$ passes through zero or not, we obtain the formula of Shafranov's work ^{1,*}.

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CITED LITERATURE

¹ V. D. Shafranov, *Plasma Physics and the Problem of Controlled Thermonuclear Reactions*, 4, Publishing House of the Academy of Sciences of the USSR, 1958, p. 61.

² B. R. Suydam, Proc. II International Conference on the Peaceful Uses of Atomic Energy, Geneva, 1958, 1, *Physics of Hot Plasma and Thermonuclear Reactions*, 1959, p. 89.

* Integrating (2)–(3), one can show that formula (12) remains valid for an arbitrary distribution of the field $H(r)$, if $d/dr \gg 1/r$, $\Omega^2 > 0$, and $|\Omega|$ is not close to zero.

Note: Figure translations are in progress. See original paper for figures.

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