

Bifurcations of Simple Double Points and Cusps of Real Plane Algebraic Curves

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Abstract

Full Text

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(Presented by Academician I. G. Petrovskii, 20 X 1961)

Mathematics

1. We shall adopt the following notation and terminology: $(x : y : z)$ are the coordinates of a point in the complex projective plane $*R^2$; R^2 is the real projective plane; $R^2 \subset *R^2$; $*E^2$ is the affine complex plane obtained from $*R^2$ by setting $z = 1$; E^2 is the real affine plane belonging to $*E^2$. In $*E^2$ a unitary metric is fixed and, consequently, in E^2 a Euclidean one. $\rho^*(a^0)$ denotes the ρ -neighborhood of a point $a^0 \in *E^2$ in $*E^2$, and $\rho(a^0)$ the ρ -neighborhood of a point $a^0 \in E^2$ in E^2 .

It is known that to every real algebraic curve of order m , given by the equation

$$H(x, y, z) \equiv \sum_{\alpha+\beta+\gamma=m} A_{\alpha\beta} x^\alpha y^\beta z^\gamma = 0 \quad (1)$$

(where $A_{\alpha\beta}$ is a nonzero system of real numbers; α, β, γ are nonnegative integers), there corresponds one-to-one the ratio of the coefficients of this curve. The set of such ratios for all real curves of order m is the real projective space

$$R^N \left(N = \frac{m(m+3)}{2} \right).$$

The terms "point" and "curve" in R^N will be taken in quotation marks, in distinction from the same terms in $*R^2$. We shall also call the real curve H a "point" H in the space R^N . Let $E_{\alpha_0\beta_0}^N$ denote the real affine space obtained from R^N if the coefficient $A_{\alpha_0\beta_0}$ is set equal to one. In $E_{\alpha_0\beta_0}^N$ we introduce a Euclidean metric. $S(F, \varepsilon)$ denotes the ε -neighborhood of a point $F \in E_{\alpha_0\beta_0}^N$ in $E_{\alpha_0\beta_0}^N$.

A set Q^n of "points" in some n -dimensional Euclidean space $E^n(\tau_1, \tau_2, \dots, \tau_n)$, defined by the conditions

$$|\tau_j| \leq v \quad (j = 1, 2, \dots, n), \quad (2)$$

where $v > 0$ is a constant, is called an n -dimensional element. A subset in Q^n defined by the inequality $\tau_j > 0$ ($\tau_i < 0$) for some j will be called a semielement and denoted by ${}^+Q^n$ (${}^-Q^n$).

A set of "points" in $E_{\alpha_0\beta_0}^N$ homeomorphic to Q^n will be denoted by P^n and will be called an n -dimensional element in R^N , while a set homeomorphic to ${}^+Q^n$ (${}^-Q^n$) will be denoted by ${}^+P^n$ (${}^-P^n$) and will be called a semielement in R^N . If the "point" F is an interior "point" for P^n , we shall write $P^n(F)$. If $P^n(F)$ is given in such a way that all coefficients $A_{\alpha\beta}$ are expressed in terms of n of them by power series with real coefficients convergent in $P^n(E)$, then $P^n(F)$ will be called an analytic element.

2. Let

$$F \equiv \sum_{\alpha+\beta+\gamma=m} A_{\alpha\beta}^0 x^\alpha y^\beta z^\gamma = 0$$

be a given curve of order m , having the following singular points:

1°. δ' simple real double points b_s ($s = 1, 2, \dots, \delta'$).

2°. δ'' pairs of imaginary conjugate simple double points a_ν and $\bar{a}_\nu$ ($\nu = 1, 2, \dots, \delta''$).

In all, there are $\delta' + 2\delta'' = \delta$ simple double points.

3°. k' real cusps d_μ ($\mu = 1, 2, \dots, k'$).

4°. k'' pairs of imaginary conjugate cusps e_σ and $\bar{e}_\sigma$ ($\sigma = 1, 2, \dots, k''$).

Altogether $k' + 2k'' = k$ cusps. F has no other singular points in ${}^*R^2$.

Theorem 1*. *There exist such ${}^*E^2$ and $E_{\alpha_0\beta_0}^N$ that all singular points of the curve F lie in ${}^*E^2$, $F \in E_{\alpha_0\beta_0}^N$, and for every sufficiently small $\rho > 0$ there exists an $\varepsilon > 0$ such that:*

1°. Every curve $H \in S(F, \varepsilon) \subset E_{\alpha_0\beta_0}^N$ can have singular points only in the ρ^* -neighborhoods of the singular points of the curve F .

2°. Every curve $H \in S(F, \varepsilon)$ can have in $\rho^*(b_s)$ at most one singular point, which can only be an ordinary real double point. There exists an element ${}^{**}P_{b_s}^N(F) \in S(F, \varepsilon)$ such that the set of curves $H \in P_{b_s}^N$ having in $\rho(b_s)$ a singular point is an analytic element $P_{b_s}^{(N-1)}(F)$, dividing the element $P_{b_s}^N$ into two semi-elements ${}^+P_{b_s}^N$ and ${}^-P_{b_s}^N$. The tangent hyperplane to $P_{b_s}^{(N-1)}$ at the "point" F has equation

$$H(b_s) = 0. \quad (3)$$

3°. Every curve $H \in S(F, \varepsilon)$ can have in $\rho^*(a_\nu)$ at most one singular point, which can only be an ordinary imaginary double point, and in this case the

curve H has in $\rho^*(\bar{a}_\nu)$ the imaginary conjugate ordinary double point. There exists an element $P_{a_\nu}^N(F) \in S(F, \varepsilon)$ in which the curves H having a pair of imaginary conjugate singular points in $\rho^*(a_\nu)$ and $\rho^*(\bar{a}_\nu)$ form an analytic element $P_{a_\nu}^{(N-2)}(F)$ with tangent plane at the “point” F determined by the equations

$$\frac{1}{2} [H(a_\nu) + H(\bar{a}_\nu)] = 0, \quad \frac{1}{2i} [H(a_\nu) - H(\bar{a}_\nu)] = 0. \quad (4)$$

4°. Every curve $H \in S(F, \varepsilon)$ can have in $\rho^*(d_\mu)$ at most one singular point, which can only be real and either an ordinary double point or a cusp. There exists an element $P_{d_\mu}^N(F)$ in which there is an element $P_{d_\mu}^{(N-1)}(F)$ consisting of the curves H having a singular point in $\rho(d_\mu)$. The curves H in $P_{d_\mu}^N$ that have a cusp in $\rho(d_\mu)$ form an analytic element $P_{d_\mu}^{(N-2)}(F)$, dividing $P_{d_\mu}^{(N-1)}$ into two semi-elements: $+P_{d_\mu}^{(N-1)}$ (a curve $H \in +P_{d_\mu}^{(N-1)}$ has an isolated singular point in $\rho(d_\mu)$), and $-P_{d_\mu}^{(N-1)}$ (a curve $H \in -P_{d_\mu}^{(N-1)}$ has a node in $\rho(d_\mu)$). The tangent plane to $P_{d_\mu}^{(N-2)}$ at the “point” F has equations

$$H(d_\mu) = 0, \quad K_\mu(A_{\alpha\beta}, d_\mu) \equiv F_{yy}(d_\mu)H_x(d_\mu) - F_{xy}(d_\mu)H_y(d_\mu) = 0. \quad (5)$$

5°. Every curve $H \in S(F, \varepsilon)$ can have in $\rho^*(e_\sigma)$ at most one singular point, which can only be imaginary and either an ordinary double point or a cusp. Together with the singular point in $\rho^*(e_\sigma)$, the curve H also has the imaginary conjugate singular point of the same type in $\rho^*(\bar{e}_\sigma)$. There exists an element $P_{e_\sigma}^N(F) \in S(F, \varepsilon)$, in which there is an element $P_{e_\sigma}^{(N-2)}(F)$, consisting of the curves H having a singular point in $\rho^*(e_\sigma)$. The curves H in $P_{e_\sigma}^N$,

* Theorem 1 is a refinement and generalization (to curves with cusps) of Theorem 1 of (*). When writing (4) we were not acquainted with work (2), and in Theorems 1, 2, 5, 6 (4) repeated some results of Brusotti, as Golafassi pointed out to us (5). We take this opportunity to express our gratitude to Golafassi.

** The sign b_s on the element $P^N(F)$ will be used to distinguish elements related to different singular points of the curve F .

having a cusp at $P^*(e_\sigma)$, form an analytic element $P_{e_\sigma}^{(N-4)}(F) \subset P_{e_\sigma}^{(N-2)}$ with tangent plane defined by the equations

$$\begin{aligned} \frac{1}{2} [H(e_\sigma) + H(\bar{e}_\sigma)] &= 0, & \frac{1}{2i} [H(e_\sigma) - H(\bar{e}_\sigma)] &= 0, \\ \frac{1}{2} [K_\sigma(A_{\alpha\beta}, e_\sigma) + K_\sigma(A_{\alpha\beta}, \bar{e}_\sigma)] &= 0, & \frac{1}{2i} [K_\sigma(A_{\alpha\beta}, e_\sigma) - K_\sigma(A_{\alpha\beta}, \bar{e}_\sigma)] &= 0. \end{aligned} \quad (6)$$

Lemma 1. Let the curve F decompose into components F_h ($h = 1, 2, \dots, q$). Let $m_h \geq 1$ be the order and k_h the number of cusps of the component F_h . If the conditions

$$k_h \leq 3m_h - 1 \quad (h = 1, 2, \dots, q), \quad (7)$$

are satisfied, then the system of $(\delta + 2k)$ linear equations (with respect to all coefficients $A_{\alpha\beta}$ of the curve H), consisting of equations (3), (4), (5), and (6) for all singular points of the curve F , is linearly independent.*

Theorem 2. If conditions (7) are satisfied for the curve F , then in the space $E_{\alpha_0\beta_0}$ defined in Theorem 1 there exists such an element $P^N(F)$, containing in each of the elements: $P_{b_s}^N$ ($s = 1, 2, \dots, \delta'$), $P_{a_\nu}^N$ ($\nu = 1, 2, \dots, \delta''$), $P_{d_\mu}^N$ ($\mu = 1, 2, \dots, k'$), $P_{e_\sigma}^N$ ($\sigma = 1, 2, \dots, k$), that:

1°. The curves $H \in P^N$, having in the P^* -neighborhoods** of the simple double points of the curve F simple double points, and in the P^* -neighborhoods of the cusps of the curve F cusps, form an analytic element $P^{N-(\delta+2k)}(F)$, having at the “point” F a tangent plane defined by the system of $(\delta + 2k)$ equations: (3), (4), (5), and (6) for all singular points of the curve F .

2°. There exists a curve $G \in P^N$ and a continuous “curve” $FG \subset P^N$ (the “curve” FG joins the “points” F and G) such that:

- a) for each of the points b_s ($s = 1, 2, \dots, \delta'$) (independently of one another), at our choice either $FG \subset P_{b_s}^{N-1}$, or $FG \subset +P_{b_s}^N$ (except for the “point” F), or $FG \subset -P_{b_s}^N$ (except for the “point” F);
- b) for each of the points a_ν ($\nu = 1, 2, \dots, \delta''$) (independently of one another), at our choice either $FG \subset P_{a_\nu}^{N-2}$, or $FG \subset P_{a_\nu}^N \setminus P_{a_\nu}^{N-2}$ (except for the “point” F);
- c) for each of the points d_μ ($\mu = 1, 2, \dots, k'$) (independently of one another), at our choice either $FG \subset P_{d_\mu}^{(N-2)}$, or $FG \subset +P_{d_\mu}^{(N-1)}$ (except for the “point” F), or $FG \subset -P_{d_\mu}^{(N-1)}$ (except for the “point” F);
- d) for each of the points e_σ ($\sigma = 1, 2, \dots, k''$) (independently of one another), at our choice either $FG \subset P_{e_\sigma}^{(N-4)}$, or $FG \subset P_{e_\sigma}^{(N-2)} \setminus P_{e_\sigma}^{(N-4)}$ (except for the “point” F).

Moreover, if $H \in FG$, then the coordinates of all singular points of the curve H are continuous functions of the coefficients of the curve H .***

* Brusotti proved (2) a special case of this lemma, when $k = 0$.

** All singular points of the curve F lie in the plane ${}^*E^2$, defined in Theorem 1.

*** If assertion 2° of Theorem 2 holds, then we shall say that all singular points of the curve F can be bifurcated by arbitrarily small additions independently of one another.

Lefschetz stated (1) the theorem:

If an irreducible curve F has δ simple double points, then each double point imposes one condition on the coefficients of the curve F .

In fact, Lefschetz proved this assertion only for unicursal curves having only simple double points.

In the same work Lefschetz accepted without proof the postulate on singularities:

If a curve F is irreducible and has δ simple double points and k cusps (and has no other singularities), then each simple double point imposes one condition, and each cusp two conditions, on the coefficients of the curve F .

Coolidge proves (3) the validity of the “postulate on singularities” if $k \leq 3m - 1$. Like Lefschetz, Coolidge proves his theorems for complex irreducible curves and does not address the question of the independence of bifurcations of singular points.

Brusotti proved (2) a special case of Theorem 2, when $k = 0$. Our Theorem 2 is a generalization of the indicated theorem of Brusotti to curves with cusps.

Theorem 3. Let the curve F decompose into components F_h ($h = 1, 2, \dots, q$) of orders $m_h \geq 1$ and genera p_h . If the conditions

$$p_h \leq \frac{m_h + 4}{2} \quad (h = 1, 2, \dots, q), \quad (8)$$

are satisfied, then the assertion of Theorem 2 holds.

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Note: Figure translations are in progress. See original paper for figures.

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