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Abstract

Full Text

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SPACES $L_p^r(\Omega)$. EXTENSION AND EMBEDDING THEOREMS

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MATHEMATICS

1. In the preceding note ⁽²⁾, proceeding from the concept of a generalized Liouville derivative, we showed how one can extend the $W_p^{(l)}$ -classification of the function spaces of S. L. Sobolev ⁽¹⁾ to nonintegral indices of differentiation. In the case of a bounded domain $\Omega \subset E_n$, the corresponding direct construction encounters considerable difficulties. However, using the equivalence of the classes that interest us with classes of functions representable by Bessel potentials (see ⁽²⁾) and the recent results of E. Stein ⁽³⁾, one can reduce the matter to extending functions from Ω to E_n .^{*} In the present note we, first, carry out (for domains Ω with sufficiently smooth boundary) the indicated extension and, on its basis, come to the consideration of the spaces $L_p^r(\Omega)$; secondly, we obtain a further embedding theorem for the spaces $L_p^r(E_n)$. Naturally, the embedding theorems obtained for $L_p^r(E_n)$ also hold in $L_p^r(\Omega)$.

Definition. A function $f(x) = f(x_1, \dots, x_n)$ belongs to the space $L_p^r(\Omega)$, $2 \leq p < \infty$, $r \geq 0$, if it belongs to the Sobolev space $W_p^{([r])}(\Omega)$, and, for nonintegral r , also possesses finite integrals

$$\Lambda_p^r(f^{[r]}, \Omega) = \left\{ \int_{\Omega} dy \left(\int_{\Omega} \frac{|f^{[r]}(x) - f^{[r]}(y)|^2}{|x - y|^{n+2(r-[r])}} dx \right)^{p/2} \right\}^{1/p}, \quad (1)$$

where $f^{[r]}(x)$ is any of the partial derivatives of order $[r]$ of the function f .

With the introduction of the norm

$$\|f\|_{L_p^r(\Omega)} = \begin{cases} \|f\|_{W_p^r(\Omega)}, & \text{for integral } r \geq 0, \\ \|f\|_{W_p^{[r]}(\Omega)} + \sum_{[r]} \Lambda_p^r(f^{[r]}, \Omega), & \text{for nonintegral } r > 0, \end{cases} \quad (2)$$

the space $L_p^r(\Omega)$ becomes a Banach space.

Let us first prove that the present definition of $L_p^r(\Omega)$ for $\Omega = E_n$ coincides with the definition used in ⁽²⁾. Namely, there it was assumed that $f(x) \in L_p^r(E_n)$ if**

$$f(x) = c_{n,r} \int_{E_n} |x-t|^{(r-n)/2} K_{\frac{n-r}{2}}(|x-t|) \varphi(t) dt, \quad \varphi(t) \in L_p(E_n) \quad (3)$$

and as the norm of f in $L_p^r(E_n)$ the quantity $\|\varphi\|_{L_p(E_n)}$ was taken.

Theorem 1. For $2 \leq p < \infty$, a function $f(x)$ is representable in the form (3) if and only if the inequalities

$$B_{p,r} \|\varphi\|_{L_p(E_n)} \leq \|f\|_{L_p^r(\Omega)} \leq A_{p,r} \|\varphi\|_{L_p(E_n)} \quad (\Omega = E_n!) \quad (4)$$

hold.

* In doing so we take into account the experience of constructing the $H_p^{(r)}$ -classes of S. M. Nikol'skii ⁽⁴⁾ and the generalized (after L. N. Slobodetskii) spaces $W_p^{(r)}$ of S. L. Sobolev.

** $K_{\frac{n-r}{2}}$ is the Macdonald function; $c_{n,r} |x|^{(r-n)/2} K_{\frac{n-r}{2}}(|x|) \equiv G_r(x)$.

For $0 < r < 1$ the assertion stated is part of a theorem of E. Stein ⁽³⁾; for integral r it follows, for example, from the arguments of the note ⁽²⁾. Now let $r > 1$ be a non-integral number and let f be representable in the form (3). We use

Remark 1. The operation of differentiation maps the space $L_p^r(E_n)$ into $L_p^{r-1}(E_n)$ (and this embedding is continuous).

By the embedding theorems ⁽²⁾ and on the basis of this remark, the function f and the derivatives $f^{[r]}$ are estimated through φ in the norm L_p . Moreover, $f^{[r]} \in L_p^{r-[r]}(E_n)$ and, consequently, is representable by a Bessel potential of the form (3)

$$f^{[r]} = \int_{E_n} G_{r-[r]}(x-t) g_{[r]}(t) dt, \quad g_{[r]} \in L_p(E_n),$$

where $\|g_{[r]}\|_{L_p} \leq c \|\varphi\|_{L_p}$. Since $0 < r - [r] < 1$, by the above-mentioned Stein theorem,

$$\Lambda_p^r(f^{[r]}, E_n) \leq A \|g_{[r]}\|_{L_p}.$$

Finally we have

$$\|f\|_{W_p^{[r]}(E_n)} + \Lambda_p^r(f^{[r]}, E_n) \leq c\|\varphi\|_{L_p(E_n)},$$

which proves the right-hand part of inequality (4).

Let us prove the left embedding (4). From the finiteness of $\|f\|_{L_p^r(\Omega)}$ it follows, first, that $f \in W_p^{[r]}(E_n)$, and, second, that $\Lambda_p^r(f^{[r]}, E_n)$ is finite. By Stein's theorem the latter means that $f^{[r]} \in L_p^{r-[r]}(E_n)$. Thus, we have

$$(1 + |\lambda|^2)^{[r]/2} \tilde{f} = \tilde{g}, \quad \|g\|_{L_p} \leq c\|f\|_{W_p^{[r]}(E_n)},$$

$$(1 + |\lambda|^2)^{\frac{r-[r]}{2}} \widetilde{f^{[r]}} = (1 + |\lambda|^2)^{\frac{r-[r]}{2}} (i\lambda)^{[r]} \tilde{f} = \tilde{g}_{[r]}, \quad \|g_{[r]}\|_{L_p} \leq c\Lambda_p^r(f^{[r]}, E_n).$$

Using the linearity of the Fourier transform and the theorem on multipliers, we obtain from this

$$(1 + |\lambda|^2)^{r/2} \tilde{f} = \tilde{\varphi},$$

where the function φ is represented in the form of a linear combination of functions each of which is estimated through g or $g_{[r]}$ in the norm $L_p(E_n)$, i.e., in the final result, we obtain the assertion completing the proof of the theorem:

$$\|\varphi\|_{L_p(E_n)} \leq c[\|f\|_{W_p^{[r]}(E_n)} + \Lambda_p^r(f^{[r]}, E_n)].$$

Here it is appropriate to say that the exceptional status of integral r in the definition of $L_p^r(\Omega)$ could have been avoided by using second differences of f (see (3)). Theorem 1 shows that the integer-valued (with respect to r) classes $L_p^r(\Omega) \equiv L_p^r(E_n)$ are not exceptional in the functional scale under consideration.

2. We now prove the extension theorem, which in essence justifies considering the spaces $L_p^r(\Omega)$ for $\Omega \neq E_n$ in the definition given above.

Theorem 2. *If the boundary Γ of the domain Ω belongs to the class $C^{[r]+1}$, then any function $f \in L_p^r(\Omega)$ can be extended to the whole space E_n with preservation of the class, i.e., for the extended function F the inequality holds*

$$\|F\|_{L_p^r(E_n)} \leq c\|f\|_{L_p^r(\Omega)},$$

where the constant c does not depend on f .

In the case of integral r the theorem was proved in ⁽⁴⁾ (see also ⁽⁵⁾). For non-integral r we shall carry out the calculations in the simplest case, when Ω coincides with the upper half-space E_n^+ ($x_n > 0$). Define the extended function as follows:

$$F(x) = \begin{cases} f(x), & \text{for } x_n > 0 \ (x \in E_n^+), \\ \sum_{k=1}^{[r]+1} \lambda_k f\left(x_1, \dots, x_{n-1}, -\frac{1}{k}x_n\right), & \text{for } x_n < 0 \ (x \in E_n^-). \end{cases}$$

where the numbers λ_k are chosen from the conditions

$$(-1)^l \lambda_1 + \left(-\frac{1}{2}\right)^l \lambda_2 + \dots + \left(-\frac{1}{[r]+1}\right)^l \lambda_{[r]+1} = 1, \quad l = 0, \dots, [r].$$

It is clear that the function $F(x) \in W_p^{[r]}(E_n)$ (see ^(4,5)), and it remains only to verify the finiteness of the expressions

$$[\Lambda_p^r(F^{[r]}, E_n)]^p \equiv [\Lambda]^p = \int_{E_n} dy \left(\int_{E_n} \frac{|F^{[r]}(x) - F^{[r]}(y)|^2}{|x - y|^{n+2(r-[r])}} dx \right)^{\frac{p}{2}}.$$

We have, obviously,

$$[\Lambda]^p \leq 2^{\frac{p}{2}-1} \left\{ \int_{E_n^+} dy \left(\int_{E_n^+} \dots dx \right)^{\frac{p}{2}} + \int_{E_n^-} dy \left(\int_{E_n^+} \dots dx \right)^{\frac{p}{2}} + \int_{E_n^+} dy \left(\int_{E_n^-} \dots dx \right)^{\frac{p}{2}} + \int_{E_n^-} dy \left(\int_{E_n^-} \dots dx \right)^{\frac{p}{2}} \right\}. \quad (5)$$

The first of the integrals in braces exists by assumption—it is $[\Lambda_p^r(F^{[r]}, E_n^+)]^p$. Let us show, for example, the estimate of the second integral:

$$\begin{aligned} & \int_{E_n^-} dy \left(\int_{E_n^+} \frac{\left| \frac{\partial^{[r]} F(x)}{\partial x_1^{l_1} \dots \partial x_n^{l_n}} - \frac{\partial^{[r]} F(y)}{\partial y_1^{l_1} \dots \partial y_n^{l_n}} \right|^2}{|x - y|^{n+2(r-[r])}} dx \right)^{\frac{p}{2}} = \\ & = \int_{E_n^-} dy \left(\int_{E_n^+} \frac{\left| \sum \lambda_k \left(-\frac{1}{k}\right)^{l_n} [f^{[r]}(x_1, \dots, x_n) - f^{[r]}(y_1, \dots, y_{n-1}, -\frac{1}{k}y_n)] \right|^2}{|x - y|^{n+2(r-[r])}} dx \right)^{\frac{p}{2}}. \end{aligned}$$

We have used the fact that

$$\sum_{k=1}^{[r]+1} \lambda_k \left(-\frac{1}{k}\right)^{l_n} = 1.$$

Applying further Hölder's inequality for sums and making the change of variables $\eta_1 = y_1, \dots, \eta_{n-1} = y_{n-1}$, $\eta_n = -\frac{1}{k}y_n$, we obtain

$$\begin{aligned} &\leq ([r] + 1)^{\frac{p}{2}} \sum_{k=1}^{[r]+1} |\lambda_k|^p k^{\frac{p}{2} - pl_n} \times \\ &\times \int_{E_n^+} d\eta \left(\int_{E_n^+} \frac{|f^{[r]}(x_1, \dots, x_n) - f^{[r]}(\eta_1, \dots, \eta_n)|^2 dx}{[(x_1 - \eta_1)^2 + \dots + (x_{n-1} - \eta_{n-1})^2 + (x_n + k\eta_n)^2]^{n/2 + r - [r]}} \right)^{\frac{p}{2}} < \\ &< c \int_{E_n^+} d\eta \left(\int_{E_n^+} \frac{|f^{[r]}(x_1, \dots, x_n) - f^{[r]}(\eta_1, \dots, \eta_n)|^2 dx}{|x - \eta|^{n+2(r-[r])}} \right)^{\frac{p}{2}}. \end{aligned}$$

The estimates of the remaining integrals on the right in (5) proceed analogously, and the theorem for $\Omega = E_n^+$ may be considered proved.

It follows from the computations given how one can (by resorting to a sufficiently regular transformation of the domain Ω) extend a function $f \in L_p^r(\Omega)$ beyond the domain locally in a neighborhood of each boundary point. By virtue of the results of Whitney and Hestenes, this guarantees the extendability of f to the whole space E_n . The theorem is proved.

Theorem 2 makes it possible to carry over the results of Note (2) to the case of a domain Ω with sufficiently smooth boundary; namely, the following is valid:

Theorem 3. The function $f(x) \in L_p^r(\Omega)$ on any sufficiently smooth manifold $S \subset \bar{\Omega}$ of dimension m ($1 \leq m \leq n$) belongs to the space $L_{p'}^{r'}(S)$, where the parameters r, p, n, r', p', m are related by

$$2 \leq p < p' < \infty, \quad r - \frac{n}{p} \geq r' - \frac{m}{p'}, \quad 0 \leq r' < r$$

(if $m = n$, the equality $p = p'$ is also allowed). The indicated embedding

$$L_p^r(\Omega) \subset L_{p'}^{r'}(S)$$

is continuous.

Remark 2. The space $L_{p'}^{r'}(S)$ for $m \neq n$, in the case where the manifold S is not flat, is understood as follows. It is assumed that the manifold S admits a finite

covering by “simple” manifolds, each of which is sufficiently smoothly mapped onto a domain of the Euclidean space E_m . If the function $f(x)$, considered on the manifold S , under such a mapping is a function of class $L_{p'}^{r'}$ in the Cartesian image of each element of the covering, then we say that

$$f(x)|_S \in L_{p'}^{r'}(S).$$

However, one may also use the invariant notation of the integral $\Lambda_p^r(f^{[r]}, S)$, without resorting to a covering of S .

Remark 3. Since the derivatives of f normal to S (for $m \neq n$) are expressed in terms of partial derivatives, and for the latter Remark 1 applies, it follows from Theorem 3 and this remark that we also obtain statements concerning normal derivatives.

3. In conclusion we shall give statements on the properties of functions from

$$L_p^r(E_n) \text{ for } rp > n \left(1 < p < \infty, \alpha = r - \frac{n}{p} - \left[r - \frac{n}{p} \right] \right).$$

Theorem 4. From $f(x) \in L_p^r(E_n)$ it follows that

$$\Delta_1 \left(f^{[r-\frac{n}{p}]}, h \right) = \left| f^{[r-\frac{n}{p}]}(x+h) - f^{[r-\frac{n}{p}]}(x) \right| = o(|h|^\alpha) \quad \text{for noninteger } r - \frac{n}{p},$$

$$\Delta_2 \left(f^{r-\frac{n}{p}-1}, h \right) = \left| f^{r-\frac{n}{p}-1}(x+h) - 2f^{r-\frac{n}{p}-1}(x) + f^{r-\frac{n}{p}-1}(x-h) \right| = o(|h|)$$

for integer $r - \frac{n}{p} \geq 1$.

Some further information about the quality of the function $o(t)$ in Theorem 4 is given by

Theorem 5. From $f(x) \in L_p^r(E_n)$ it follows that

$$\int_{E_n} \frac{|\Delta_1 \left(f^{[r-\frac{n}{p}]}, h \right)|^p}{|h|^{n+p\alpha}} dh \leq c \|f\|_{L_p^r(E_n)}^p, \quad \text{for noninteger } r - \frac{n}{p},$$

$$\int_{E_n} \frac{|\Delta_2 \left(f^{r-\frac{n}{p}-1}, h \right)|^p}{|h|^{n+p}} dh \leq c \|f\|_{L_p^r(E_n)}^p, \quad \text{for integer } r - \frac{n}{p} \geq 1.$$

For integer $r = l$, and also for $f \in L_2^r(E_n)$, Theorem 4, for integer r and $2 \leq p < \infty$, and Theorem 5 follow from embeddings obtained in the work ⁽⁶⁾. Our proofs are based on the representation of f by a Bessel potential and on various estimates of the kernel $G_r(x)$.

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Note: Figure translations are in progress. See original paper for figures.

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