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Abstract

Full Text

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MATHEMATICS

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ON THE FUNDAMENTAL SOLUTION OF A PARABOLIC EQUATION

(Presented by Academician I. G. Petrovskii, 18 VI 1962)

It is known that for the parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad x = (x_1, x_2, \dots, x_n) \in E_n \quad (1)$$

with sufficiently smooth bounded coefficients there exists a fundamental solution $G(t, \tau, x, \xi)$, satisfying equation (1) in the variables t, x . The solution $u_\tau(t, x)$ of equation (1) with the initial condition

$$u_\tau(\tau, x) = \varphi(x) \quad (2)$$

is represented in the form

$$u_\tau(t, x) = \int_{E_n} G(t, \tau, x, \xi) \varphi(\xi) d\xi. \quad (3)$$

Among the other properties of the function $G(t, \tau, x, \xi)$, we note here only that $0 < G(t, \tau, x, \xi) < K(\delta)$ if $t - \tau > \delta$. Then from formula (3) it follows that

$$|u_\tau(t, x)| \leq K(\delta) \int_{E_n} |\varphi(\xi)| d\xi \quad \text{for } t - \tau > \delta > 0. \quad (4)$$

In the work [1], under rather weak restrictions on the coefficients, the existence of a fundamental solution was proved for equation (1) and for the more general case of systems parabolic in the sense of I. G. Petrovskii. For this it is sufficient that the coefficients be bounded, continuous in t uniformly with respect to x , and that the Hölder condition be satisfied with respect to the spatial variables $x_1, x_2, \dots, x_n$.

We shall show that the latter condition is essential. If one requires only continuity of the coefficients of equation (1), then, generally speaking, one cannot expect representability of the solution of problem (1), (2) in the form (3) with a function $G(t, \tau, x, \xi)$ bounded only for $t - \tau > \delta > 0$.

More precisely, we shall construct the equation

$$\frac{\partial u}{\partial t} = a(t, x) \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq t \leq 1, \quad (5)$$

with a continuous function $a(t, x)$,

$$\frac{1}{2} \leq a(t, x) \leq \frac{3}{2}, \quad (6)$$

such that for the solutions of problem (5), (2) (if they exist), generally speaking, relation (4) is false.

Put $a(t, x) = 1 + b(t, x)$. Let γ be a positive number less than $1/4$. Define $b(t, x)$ for sufficiently small t such that,

$$\left(\ln \frac{1}{t} \right)^\gamma > 4,$$

in the following way:

$$b(t, x) = \begin{cases} -\frac{1}{\left(\ln \frac{1}{t} \right)^\gamma}, & \text{when } 0 \leq \frac{x^2}{t} \leq 2 - \frac{2}{\left(\ln \frac{1}{t} \right)^{2\gamma}}, \\ 0, & \text{when } 2 - \frac{1}{\left(\ln \frac{1}{t} \right)^{2\gamma}} \leq \frac{x^2}{t} \leq 2, \\ \frac{1}{\left(\ln \frac{1}{t} \right)^\gamma}, & \text{when } \frac{x^2}{t} \geq 4. \end{cases}$$

For the remaining t and x we extend $b(t, x)$ evenly with respect to x , nondecreasing in x for $x > 0$, so that $b(t, x)$ is smooth for $t > 0$ and condition (6) is satisfied. It is obvious that $a(t, x)$ is continuous for $t \geq 0$, but satisfies not only the Hölder condition, but also the Dini condition.

Below a function $v(t, x)$, smooth for $t > 0$, will be constructed such that

$$Lv \equiv a(t, x) \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} > 0 \quad \text{when } 0 < t < t_0, \quad (7)$$

and

$$\int_{-\infty}^{\infty} v(t, x) dx \rightarrow 0.$$

By the maximum principle the solution of the Cauchy problem

$$Lu_\tau = 0; \quad u_\tau(\tau, x) = v(\tau, x)$$

satisfies the inequality $u_\tau(t, x) > v(t, x)$ for $\tau < t < t_0$. Consequently,

$$\frac{u_\tau(t, x)}{\int_{-\infty}^{\infty} u_\tau(\tau, x) dx} \xrightarrow{\tau \rightarrow 0} \infty,$$

and relation (4) is thereby not fulfilled.

We first construct an auxiliary even function $\psi(z)$. Let

$$\psi(z) = 12 - z^2 + z^4$$

for $|z| \leq 1/2$. Further, $\psi(z)$ is extended smoothly so that $z\psi'(z) \leq 0$ and $\psi(z) = 10$ for $|z| \geq 1$.

Put

$$v(t, x) = \frac{1}{\sqrt{t} \ln \frac{1}{t}} \psi \left(\frac{x}{\sqrt{t} \left(\ln \frac{1}{t} \right)^\gamma} \right) e^{-x^2/4t}.$$

For compactness of notation, denote

$$\frac{1}{\ln \frac{1}{t}} = \alpha(t) \quad \text{and} \quad \frac{x}{\sqrt{t}} = \xi.$$

It is easy to verify that

$$\frac{tLv}{v} = b \frac{\xi^2 - 2}{4} + \frac{1}{\psi} \left[b\alpha^{2\gamma}\psi''(\xi\alpha^\gamma) - b\xi\alpha^\gamma\psi'(\xi\alpha^\gamma) + \alpha^{2\gamma}\psi''(\xi\alpha^\gamma) - \frac{1}{2}\xi\alpha^\gamma\psi'(\xi\alpha^\gamma) - \gamma\xi\alpha^{\gamma+1}\psi'(\xi\alpha^\gamma) \right] - \alpha.$$

We shall prove the validity of inequality (7) for some $t_0 > 0$.

If $\xi^2 \geq 4$, then

$$\frac{tLv}{v} \geq \frac{\alpha^\gamma}{2} - M_1\alpha^{2\gamma} > 0$$

for sufficiently small t (constants will be denoted by M_i).

If $\xi^2 \leq 4$, then $|\xi| \leq \frac{1}{2\alpha^\gamma}$ for sufficiently small t , and, consequently,

$$\psi(z) = 12 - z^2 + z^4.$$

Then

$$\frac{tLv}{v} = b \frac{\xi^2 - 2}{4} + \frac{1}{\psi} [2b\alpha^{2\gamma}(\xi^2 - 1) + 4b\xi^2\alpha^{4\gamma}(3 - \xi^2) + \alpha^{2\gamma}(\xi^2 - 2) + 2\xi^2\alpha^{4\gamma}(6 - \xi^2) + 2\gamma\xi^2\alpha^{2\gamma+1} - 4\gamma\xi^4\alpha^{4\gamma+1}]$$

Further estimates will be carried out for sufficiently small t without special reservations.

If $2 \leq \xi^2 \leq 4$, then $0 \leq b \leq \alpha^\gamma$ and

$$\frac{tLv}{v} \geq \frac{1}{\psi} (8\alpha^{4\gamma} - M_2\alpha^{5\gamma}) - \alpha > 0.$$

If $2 - \alpha^{2\gamma} \leq \xi^2 \leq 2$, then $b \equiv 0$ and

$$\frac{tLv}{v} \geq \frac{1}{\psi} [-\alpha^{4\gamma} + 4\alpha^{4\gamma}(6 - \xi^2) - M_3\alpha^{6\gamma}] - \alpha > 0.$$

For $2 - 2\alpha^{2\gamma} \leq \xi^2 \leq 2 - \alpha^{2\gamma}$, the function $b \leq 0$ and

$$\begin{aligned} \frac{tLv}{v} \geq b \left[\frac{\xi^2 - 2}{4} + \frac{2\alpha^{2\gamma}(\xi^2 - 1)}{\psi} + \frac{4\xi^2\alpha^{4\gamma}(3 - \xi^2)}{\psi} \right] + \\ + \frac{1}{\psi} [-2\alpha^{4\gamma} + 4\alpha^{4\gamma}(6 - \xi^2) - M_4\alpha^{6\gamma}] - \alpha > -b \left(\frac{\alpha^{2\gamma}}{4} - \frac{2\alpha^{2\gamma}}{\psi} - M_2\alpha^{4\gamma} \right) > 0, \end{aligned}$$

since $\psi \geq 10$.

If $2 - \alpha^\gamma \leq \xi^2 \leq 2 - \alpha^{2\gamma}$, then $b = -\alpha^\gamma$ and

$$\frac{tLv}{v} \geq -\alpha^\gamma \left(-\frac{\alpha^{2\gamma}}{2} + \frac{2\alpha^{2\gamma}}{\psi} + M_6\alpha^{4\gamma} \right) + \frac{1}{\psi} (-\alpha^{3\gamma} - M_7\alpha) - \alpha > 0.$$

If $\xi^2 \leq 2 - \alpha^\gamma$, then

$$\frac{tLv}{v} > \frac{\alpha^{2\gamma}}{4} - \frac{2\alpha^{2\gamma}}{\psi} - M_8\alpha^{3\gamma} > 0.$$

Thus the validity of inequality (7) has been proved and, consequently, the absence for equation (5) of a fundamental solution with the usual properties. We note that the equation $\frac{\partial u}{\partial t} = a(x) \frac{\partial^2 u}{\partial x^2}$ has a fundamental solution under the sole condition of continuity of the function $a(x)$ (2).

However, one may expect that the solution of problem (1), (2) exists for continuous bounded coefficients and is representable in the form

$$u_\tau(t, x) = \int \varphi(\xi) P(t, \tau, x, d\xi),$$

where the measure $P(t, \tau, x, d\xi)$ has a finite density $G(t, \tau, x, \xi)$, if the coefficients satisfy Hölder's condition. In the example of equation (5) constructed by us, this density apparently tends to infinity as $\tau \rightarrow 0$, $\xi \rightarrow 0$, but more slowly than $\xi^{-\beta}$, for any $\beta > 0$.

At least, it is easy to prove that the solution of equation (1), which is equal to one for $t = 0$, $|x| \leq r_0$, and zero for $t = 0$, $|x| > r_0$, satisfies the inequality

$$|u(x, t)| < Mr_0^{n-\beta}.$$

Here $\beta > 0$ is arbitrary, and the constant M depends only on β and t . The same estimate is preserved in the case when lower-order terms with bounded coefficients are present in the equation.

We note that the phenomenon described above for a parabolic equation also occurs for a second-order elliptic equation with discontinuous coefficients. It is known that in the case of smooth coefficients, for solutions of the equation

$$a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} = 0, \quad ac - b^2 \geq \mu > 0,$$

the inequality holds

$$|u(P)| \leq M \int_{C_\rho} |u(Q)| dS, \quad (8)$$

where C_ρ is the circle with center at the point P and radius ρ ($\rho_0 < \rho < \rho_1$), and the constant M depends only on ρ_0 and ρ_1 . However, for solutions of the equation

$$Lu \equiv (1 + \gamma(r) \cos^2 \theta)u_{xx} + \gamma(r) \sin 2\theta u_{xy} + (1 + \gamma(r) \sin^2 \theta)u_{yy} = 0,$$

where $\gamma(r) = \frac{2}{\ln \frac{r}{50}}$, $r \leq 2$ is the polar radius and θ the polar angle, inequality

(8), generally speaking, does not hold.

To prove this assertion it is enough, as above, to consider the function $v = \frac{2 \cos \theta - r}{r} \cdot \frac{1}{\ln \frac{50}{r}}$ and verify that $Lv \geq 0$ in the disk

$$r \leq 2 \cos \theta.$$

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