

PHYSICS

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Abstract

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KINETIC EQUATION FOR A RELATIVISTIC GAS IN AN ARBITRARY GRAVITATIONAL FIELD

(Presented by Academician V. A. Fock, 11 XII 1961)

In the present work a kinetic equation is established that determines the behavior of a relativistic ideal gas in an arbitrary Einstein gravitational field. This equation cannot be obtained by the method by which the Boltzmann equation is usually obtained in the nonrelativistic case. The very concept of a distribution function is usually based on the concept of time t , whereas in an arbitrary gravitational field synchronization of clocks is impossible.

In work ⁽¹⁾ a method was proposed that eliminates this shortcoming of the usual approach to the description of an ideal gas. It was further developed in works ^(2,3). This method makes it possible to find the kinetic equation of motion of a relativistic ideal gas in an arbitrary gravitational field.

According to this method, it is first necessary to define the state of a particle. By the state of a particle we shall mean the totality of its space-time position and momentum. If the rest mass of the particle is greater than zero, then one may give another definition of the state of a particle, equivalent to the first but more visual: the state of a particle may be called a line tangent to the space of events and lying inside the tangent light cone. The point of tangency determines the space-time position of the particle, and its place in the tangent space determines the velocity of the particle, considered without reference to any coordinate system. The differential neighborhood on this line with center at the point of tangency coincides with the corresponding neighborhood on the world trajectory of the particle.

Next we must define the space of states of the particle. We shall assume that the metric tensor $g_{\alpha\beta}(x)$, defining the interval ds ($ds^2 = g_{\alpha\beta}(x) dx^\alpha dx^\beta$), is continuous throughout the entire space of events X . By the space of states F of a particle we shall mean the skew product ^(4,5), whose base is the space X , and whose fiber is the momentum space of the particle Π . If the rest mass is greater than zero, then one may give an equivalent but more visual definition of the space of states: the space of states of a particle may be called the bundle of timelike straight lines tangent to X .

At first we shall restrict ourselves to the usual and, moreover, simplest

case, when the whole space X can be covered by a single coordinate system x^0, x^1, x^2, x^3 such that the vector field $\xi(x)$, $x \in X$, with contravariant components

$$\{1, 0, 0, 0\}$$

at each point x is directed into the interior of the upper half of the light cone. The hyperplane tangent to X at the point x and orthogonal to the vector $\xi(x)$ lies outside the light cone. It follows from this that $g_{00} > 0$, that the matrix $g_{0i}g_{0k} - g_{00}g_{ik}$ is positive definite, and that as coordinates in the space of states F one may choose the seven quantities $x^0, x^1, x^2, x^3, p^1, p^2, p^3$, where p^i are the contravariant components of the particle momentum, lying in the range $-\infty < p^1, p^2, p^3 < \infty$. Since

$g_{\alpha\beta}p^\alpha p^\beta = m^2c^2$, where m is the rest mass of the particle, the zeroth contravariant component of the momentum is equal to

$$p^0 = \frac{p_0 - g_{0i}p^i}{g_{00}}, \quad (1)$$

where the covariant component of the momentum p_0 is equal to

$$p_0 = \sqrt{m^2c^2g_{00}(x) + [g_{0i}(x)g_{0k}(x) - g_{00}(x)g_{ik}(x)]p^ip^k}. \quad (2)$$

The positive sign in front of the root in formula (2) corresponds to the fact that the vector $\xi(x)$ is directed into the interior of the upper half of the light cone.

We now dwell on the choice of the particle's proper time. The definition of the particle's proper time introduced in works ^(3,6) is convenient in that it is also suitable for particles with zero rest mass. We shall adopt this definition of proper time here as well: the interval of the particle's proper time $d\tau$ is the coefficient of proportionality between the displacement vector dx along its world trajectory and its four-dimensional momentum: $dx = P d\tau$, or, in components, $dx^\alpha = p^\alpha d\tau$. The equations of motion of gas particles in the intervals between collisions determine, in the state space F , the vector field $f(M)$, $M \in F$:

$$f^\alpha = \frac{dx^\alpha}{d\tau} = p^\alpha, \quad f^{3+k} = \frac{dp^k}{d\tau} = -\Gamma_{\beta\gamma}^k p^\beta p^\gamma, \quad (3)$$

$\Gamma_{\beta\gamma}^k$ are Christoffel symbols of the second kind.

As an element of volume in the state space F , it is convenient to choose the 7-linear skew-symmetric form $\varepsilon(M; d_0, d_1, \dots, d_6)$, $M \in F$, which takes on the vectors of elementary displacements along the coordinate lines the value

$$\varepsilon(M; d_0, d_1, \dots, d_6) = -\frac{g}{p_0} dx^0 dx^1 dx^2 dx^3 dp^1 dp^2 dp^3, \quad (4)$$

where g is the determinant of the matrix $g_{\alpha\beta}(x)$.

Indeed, let us pass to new coordinates $x^{0'}, x^{1'}, x^{2'}, x^{3'}$ in the event space X , satisfying the same conditions that were imposed on the coordinates x^0, x^1, x^2, x^3 . The Jacobian of the transformation

$$x^{\alpha'} = \varphi^\alpha(x^0, x^1, x^2, x^3), \quad p^{i'} = \sum_{\beta=0}^3 p^\beta \frac{\partial}{\partial x^\beta} \varphi^i(x^0, x^1, x^2, x^3) \quad (5)$$

is equal to

$$J = \frac{\partial(x^{0'}, x^{1'}, x^{2'}, x^{3'}, p^{1'}, p^{2'}, p^{3'})}{\partial(x^0, x^1, x^2, x^3, p^1, p^2, p^3)} = \frac{p'_0}{p_0} \left[\frac{\partial(x^{0'}, x^{1'}, x^{2'}, x^{3'})}{\partial(x^0, x^1, x^2, x^3)} \right]^2 = \frac{p'_0 g}{p_0 g'}, \quad (6)$$

so that expression (4) is invariant with respect to the coordinate transformations (5) in the state space F .

As an element of area $d\Sigma$ on a hypersurface S in the particle's state space, it is convenient to choose the 6-linear skew-symmetric form $\varepsilon(M; f(M), d_1, \dots, d_6)$, where $d_1, \dots, d_6$ are vectors of elementary displacements along this hypersurface, and $f(M)$ is the vector defined above (see (3)). In detailed notation the element $d\Sigma$ is expressed as follows:

$$d\Sigma = -\frac{g}{p_0} \begin{vmatrix} f^0 & f^1 & f^2 & f^3 & f^4 & f^5 & f^6 \\ d_1 x^0 & d_1 x^1 & d_1 x^2 & d_1 x^3 & d_1 p^1 & d_1 p^2 & d_1 p^3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_6 x^0 & d_6 x^1 & d_6 x^2 & d_6 x^3 & d_6 p^1 & d_6 p^2 & d_6 p^3 \end{vmatrix}. \quad (7)$$

It is not difficult to verify that the exterior derivative of the 6-dimensional skew form

$\varepsilon(M; f(M), d_1, \dots, d_6)$ is equal to zero, i.e.

$$\sum_{r=0}^6 \frac{\partial}{\partial x^r} \left[-\frac{g}{p_0} f^r \right] = 0, \quad (8)$$

where $x^4 = p^1$, $x^5 = p^2$, $x^6 = p^3$.

Now one can without difficulty extend the results of paper (6) to the general case of an arbitrary gravitational field. Here we shall write down the Boltzmann kinetic equation. Consider an N -component ideal gas.

Let $A_i(x, P)$ be the distribution function of the i -th component of the gas: the number of particles of species i crossing a differential area element in the space of states spanned by the vectors $d_1, \dots, d_6$ is equal to $A_i(x, P)d\Sigma$. Note that the argument P is determined by the components p^1, p^2, p^3 , since these also determine the component p^0 (see (1)). We denote

$$(P, Q) = g_{\alpha\beta} p^\alpha p^\beta, \quad \langle P, Q \rangle = \sqrt{(P, Q)^2 - (P, P)(Q, Q)},$$

$$dP = \sqrt{-g} \frac{dp^1 dp^2 dp^3}{p_0}. \quad (9)$$

Assume that the particles of the gas undergo only elastic collisions, and that the differential cross section for scattering of a particle of species i with momentum P by a particle of species j with momentum Q in the system of their center of inertia is equal to

$$\Delta\sigma = h_{ij}(\langle P, Q \rangle, \cos \vartheta) \sin \vartheta d\vartheta d\varphi. \quad (10)$$

The system of the center of inertia is considered locally, at the point $x \in X$. In an arbitrary local reference frame the differential scattering cross section can be written in the form

$d\sigma = H_{ij}(\langle P, Q \rangle, \langle P', Q \rangle, \langle P', P \rangle) dP'$, where

$$H_{ij}(\langle P, Q \rangle, \langle P', Q \rangle, \langle P', P \rangle) =$$

$$= \frac{(P + Q, P + Q)}{\langle P, Q \rangle} h_{ij} \left(\langle P, Q \rangle, 1 + \frac{(P + Q, P + Q)(P, P - P')}{\langle P, Q \rangle^2} \right) \delta((P - P', Q + P)).$$

In these notations the Boltzmann kinetic equation takes the form

$$\sum_{r=0}^6 f^r \frac{\partial}{\partial x^r} A_i(x, P) = \sum_{j=1}^N (I_{ij}^{(1)} - I_{ij}^{(2)}), \quad i = 1, 2, \dots, N, \quad (12)$$

where

$$I_{ij}^{(1)} = \int_{\Pi_i} \int_{\Pi_j} A_i(x, P') A_j(x, Q') \langle P', Q' \rangle H_{ij} \times$$

$$\times (\langle P', Q' \rangle, \langle P, Q' \rangle, \langle P, P' \rangle) dP' dQ', \quad (13)$$

$$I_{ij}^{(2)} = \int_{\Pi_i} \int_{\Pi_j} A_i(x, P) A_j(x, Q) \langle P, Q \rangle H_{ij}(\langle P, Q \rangle, \langle P', Q \rangle, \langle P', P \rangle) dP' dQ.$$

On the basis of the result of paper (7), we write the collision integral $I_{ij} = I_{ij}^{(1)} - I_{ij}^{(2)}$ of particles of species i with particles of species j in Boltzmann form

$$I_{ij} = \int_{\Pi_j} dQ \int_0^\pi \int_0^{2\pi} (A'_i A'_j - A_i A_j) \langle P, Q \rangle h_{ij}(\langle P, Q \rangle, \cos \vartheta) \sin \vartheta d\vartheta d\varphi, \quad (14)$$

where

$$A_i = A_i(x, P), \quad A_j = A_j(x, Q), \quad A'_i = A_i(x, P'), \quad A'_j = A_j(x, Q'),$$

$$P' = \frac{(P, P+Q)}{(P+Q, P+Q)}(P+Q) + \xi \sin \vartheta \cos \varphi + \eta \sin \vartheta \sin \varphi + \zeta \cos \vartheta, \quad (15)$$

$$Q' = \frac{(Q, P+Q)}{(P+Q, P+Q)}(P+Q) - \xi \sin \vartheta \cos \varphi - \eta \sin \vartheta \sin \varphi - \zeta \cos \vartheta,$$

$$\zeta = \frac{(P+Q, Q)P - (P+Q, P)Q}{(P+Q, P+Q)}, \quad (16)$$

ξ and η are some pair of vectors at the point $x \in X$, satisfying the conditions

$$(\xi, \eta) = 0, \quad (\xi, \zeta) = 0, \quad (\eta, \zeta) = 0,$$

$$(\xi, P+Q) = (\eta, P+Q) = (\zeta, P+Q) = 0, \quad (17)$$

$$(\xi, \xi) = (\eta, \eta) = (\zeta, \zeta) = -\frac{\langle P, Q \rangle^2}{(P+Q, P+Q)}.$$

Thus, the kinetic equation for a relativistic gas in an arbitrary gravitational field has been established.

We shall make a few further remarks on the case when the event space X has a more complicated topological structure and cannot be covered by a single

coordinate system. The condition of continuity of the metric tensor imposes a significant restriction on the topological structure of the event space. It is known ⁽⁵⁾ that, in order for a metric of normal hyperbolic type to exist on a differentiable manifold, it is necessary and sufficient that on this manifold there exist a continuous field $\Omega(x)$ of nonzero vectors, directed at every point x into the interior of the isotropic cone. This field can be normalized to unity at every point x . How stringent this condition is may be judged from the fact that among two-dimensional compact manifolds only the torus and the Klein bottle ⁽⁴⁾ satisfy it. The existence of such a field in the event space X means, in particular, that the 7-dimensional surface in the tangent bundle of the space X , given by the equation $(P, P) = m^2 c^2$, for $m > 0$, consists of two components. The field $\Omega(x)$ selects one component, and the field $-\Omega(x)$ the other. Either of these components may be identified with the state space of a particle with nonzero rest mass. If from the surface $(P, P) = 0$ one removes the points (x, P) , where $x \in X$, $P = 0$, then it splits into two components, and one of these components can be identified with the state space of a particle with zero rest mass. Relying on this fact and taking into account that the Jacobian ⁽⁶⁾ is positive, it is not difficult to prove the following important assertion:

The state space of a particle is orientable independently of whether or not the event space can be oriented.

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