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Abstract

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MATHEMATICS

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ON SOME CLASSES OF ANALYTIC FUNCTIONS OF TWO COMPLEX VARIABLES

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Let $D_1 \ni (0, 0)$ be a bounded, complete bicircular domain whose boundary is twice continuously differentiable and analytically convex from the outside, and such that the curve corresponding in the "absolute quadrant-plane" to the boundary of D_1 is convex or straight. As A. A. Temlyakov proved ⁽¹⁾, the boundary of this domain is parametrically given in the form

$$|w| = r_1(\tau), \quad |z| = r_2(\tau), \quad 0 \leq \tau \leq 1, \quad (1)$$

where $r_1(0) = 0$, $r_1(1) < \infty$, $0 < r_1'(\tau) \leq$ (or $\equiv$) $r_1(\tau)\tau^{-1}$ in $(0, 1]$;

$$r_2(\tau) = \exp \left[- \int \frac{\tau}{1-\tau} d \ln r_1(\tau) \right], \quad r_2(1) = 0.$$

The hypersurface (1) has the following property:

Property 1. In the "absolute quadrant-plane" the curve (1) is the envelope of the family of straight lines

$$\tau r_1^{-1}(\tau)|w| + (1-\tau)r_2^{-1}(\tau)|z| = 1, \quad 0 < \tau < 1,$$

and is situated under the envelope ⁽²⁾.

We introduce for consideration the following classes of functions.

$M_{D_1}(M_E)$ is the class of functions regular in D_1 (in the bicylinder $E\{|w| < R_1, |z| < R_2\}$) $F(w, z)$, $F(0, 0) = 1$, satisfying in D_1 (E) the condition

$$\operatorname{Re} \left(\frac{L[F(w, z)]}{F(w, z)} \right) > 0,$$

where

$$L[F(w, z)] \equiv F(w, z) + wF'_w(w, z) + zF'_z(w, z). \quad (2)$$

$N_{D_1}(N_E)$ is the class of functions $F(w, z)$ regular in $D_1(E)$, $F(0, 0) = 1$, satisfying in $D_1(E)$ the condition

$$\operatorname{Re} \left(\frac{L[L[F(w, z)]]}{L[F(w, z)]} \right) > 0.$$

The operator (2) was introduced into the study of functions of two complex variables by A. A. Temlyakov ⁽²⁾. In the present note the classes $M_{D_1}(M_E)$, $N_{D_1}(N_E)$ are studied from the point of view of estimates.

§ 1. **Theorem 1.** If the function

$$F(w, z) = \sum_{m,n=0}^{\infty} a_{mn} w^m z^n \in M_E,$$

then, for $m + n > 0$,

$$|a_{mn}| \leq (m + n + 1) R_1^{-m} R_2^{-n}.$$

Proof. Since, by the definition of the class M_E , the function $F(w, z)$ is regular in E , it follows, on the one hand, that the function $F(\rho w, \rho z)$ ($0 < \rho < 1$) is regular in the closed bicylinder $\bar{E}$ and, consequently, by virtue of A. A. Temlyakov' s integral formula ⁽³⁾,

$$a_{mn} = (4\pi^2 \rho^{m+n} R_1^m R_2^n)^{-1} \int_0^{2\pi} dt \int_0^{2\pi} \psi(\rho e^{i\varphi}, t) e^{-i[(m+n)\varphi - nt]} d\varphi, \quad (3)$$

where $\psi(\rho e^{i\varphi}, t) = F(R_1 \rho e^{i\varphi}, R_2 \rho e^{i(\varphi - t)})$; on the other hand, the function $\psi_1(\rho \zeta, t) \equiv \rho^{-1} \psi(\rho \zeta, t)$ ($0 < \rho < 1$), for any fixed value of the parameter t , $0 \leq t \leq 2\pi$, is regular in the closed disk $|\zeta| \leq 1$, and therefore, by Cauchy' s formula,

$$[(m + n + 1)!]^{-1} \psi_{1\zeta^{m+n+1}}^{(m+n+1)}(0, t) = (2\pi \rho^{m+n})^{-1} \int_0^{2\pi} \psi(\rho e^{i\varphi}, t) e^{-i[(m+n)\varphi - nt]} d\varphi. \quad (4)$$

From the fact that $F(w, z) \in M_E$, it follows that the function $\psi_1(\zeta, t)$ ($\psi_1(0, t) = 0$, $\psi'_{1\zeta}(0, t) = 1$), as a function of ζ , for any fixed value of the parameter t , $0 \leq t \leq 2\pi$, is regular in the disk $|\zeta| < 1$ and satisfies there the condition

$$\operatorname{Re} \left(\frac{\zeta \psi'_{1\zeta}(\zeta, t)}{\psi_1(\zeta, t)} \right) > 0.$$

Consequently, by (4), the function $\psi_1(\zeta, t) = \zeta + \dots$, as a function of one complex variable ζ for any fixed t , $0 \leq t \leq 2\pi$, maps the disk $|\zeta| < 1$ univalently onto a domain star-shaped with respect to the origin, and therefore (4)

$$[(m+n+1)!]^{-1} \left| \psi_{1\zeta^{m+n+1}}^{(m+n+1)}(0, t) \right| \leq m+n+1, \quad m+n=1, 2, \dots$$

Then, taking formula (4) into account, from formula (3) we find the desired estimate.

On the basis of Theorem 1 and the parametric representation of the boundary of the domain D_1 , just as Theorem 3 of the author's work (5), one proves:

Theorem 2. If the function

$$F(w, z) = \sum_{m,n=0}^{\infty} a_{mn} w^m z^n \in M_{D_1},$$

then for $m+n > 0$

$$|a_{mn}| \leq (m+n+1) r_1^{-1} \left(\frac{m}{m+n} \right) r_2^{-1} \left(\frac{m}{m+n} \right),$$

where $0^0 = 1$.

Theorem 3. If the function

$$F(w, z) = \sum_{m,n=0}^{\infty} a_{mn} w^m z^n \in N_{D_1}$$

(respectively N_E), then for $m+n > 0$

$$|a_{mn}| \leq r_1^{-m} \left(\frac{m}{m+n} \right) r_2^{-n} \left(\frac{m}{m+n} \right), \quad (5)$$

respectively

$$|a_{mn}| \leq R_1^{-m} R_2^{-n}, \quad (6)$$

where $0^0 = 1$.

Proof. It is obvious that

$$\Phi(w, z) = L[F(w, z)] = \sum_{m,n=0}^{\infty} (m+n+1)a_{mn}w^mz^n \in M_{D_1}(M_E).$$

Therefore, according to Theorem 2 (Theorem 1), we obtain (5), (6).

§ 2. **Theorem 4.** If the function $F(w, z) \in M_{D_1}$, then in D_1 we have the estimates:

$$(1+\omega)^{-2} \leq |F(w, z)| \leq (1-\omega)^{-2}, \quad (7)$$

$$\frac{1-\omega}{(1+\omega)^3} \leq |L[F(w, z)]| \leq \frac{1+\omega}{(1-\omega)^3}, \quad (8)$$

where

$$\omega = \begin{cases} |w| r_1^{-1}(1), & \text{for } z = 0, \\ |z| r_2^{-1}(0), & \text{for } w = 0, \\ \max_{0 < \tau < 1} \left[\tau \frac{|w|}{r_1(\tau)} + (1-\tau) \frac{|z|}{r_2(\tau)} \right], & \text{for the remaining points of } D_1. \end{cases}$$

We shall briefly outline the proof of the theorem. Since $F(w, z) \in M_{D_1}$, it follows from the definition of the class M_{D_1} that the function $\zeta F(r_1(\tau)\zeta, r_2(\tau)\zeta e^{-it}) = \zeta + \dots$, as a function of one complex variable ζ , for arbitrary fixed values of the parameters $\tau, t, 0 \leq \tau \leq 1, 0 \leq t \leq 2\pi$, is regular in the disk $|\zeta| < 1$ and satisfies in it the condition

$$\operatorname{Re} \left(\frac{\zeta (\zeta F(r_1(\tau)\zeta, r_2(\tau)\zeta e^{-it}))'_{\zeta}}{\zeta F(r_1(\tau)\zeta, r_2(\tau)\zeta e^{-it})} \right) > 0.$$

Consequently⁽⁴⁾, the function $\zeta F(r_1(\tau)\zeta, r_2(\tau)\zeta e^{-it}) = \zeta + \dots$, as a function of ζ , for arbitrary fixed values of τ and $t, 0 \leq \tau \leq 1, 0 \leq t \leq 2\pi$, maps the disk $|\zeta| < 1$ univalently onto a domain starlike with respect to the origin. Applying to this function the corresponding estimates for univalent functions and using, in the subsequent course of the proof, property I, we arrive at the required estimates.

Remark 1. The estimates (7), (8), in the case of domains $\{a|w| + b|z| < 1; a, b > 0\}$, are sharp, since equality in (7), (8) can occur for the function

$$F(w, z) \equiv (1 - ae^{i\alpha}w - be^{i\beta}z)^{-2} \quad (9)$$

(α and β real).

Theorem 5. If the function $F(w, z) \in N_{D_1}$, then in D_1 we have the estimates:

$$(1 + \omega)^{-1} \leq |F(w, z)| \leq (1 - \omega)^{-1}, \quad (10)$$

$$(1 + \omega)^{-2} \leq |L[F(w, z)]| \leq (1 - \omega)^{-2}. \quad (11)$$

It is proved in the same way as Theorem 4, but using the corresponding estimates for convex univalent functions in the disk $|\zeta| < 1$, since, as follows from the definition of the class N_{D_1} , the function $\zeta F(r_1(\tau)\zeta, r_2(\tau)\zeta e^{-it}) = \zeta + \dots$, as a function of one complex variable ζ , for arbitrary fixed values of the parameters τ, t , $0 \leq \tau \leq 1$, $0 \leq t \leq 2\pi$, maps the disk $|\zeta| < 1$ univalently onto a convex domain.

Remark 2. The estimates (10), (11), in the case of domains $\{a|w| + b|z| < 1; a, b > 0\}$, are sharp, since equality in (10), (11) can occur for the function

$$F(w, z) \equiv (1 - ae^{i\alpha}w - be^{i\beta}z)^{-1}. \quad (12)$$

Remark 3. Theorems 4 and 5 also hold in the case of the bicylinder E , with

$$\omega = \begin{cases} |w|R_1^{-1} & \text{for } (w, z) \in \{|w|R_1^{-1} > |z|R_2^{-1}\} \cap E, \\ |z|R_2^{-1} & \text{for } (w, z) \in \{|w|R_1^{-1} \leq |z|R_2^{-1}\} \cap E. \end{cases}$$

The estimates in Theorems 4 and 5 (the case of the bicylinder) are sharp for the set of points $\{|w|R_1^{-1} = |z|R_2^{-1}\} \cap E$, since there exist functions for which they can be attained. The form of these functions is obtained from the functions (9), (12) by replacing $ae^{i\alpha}w + be^{i\beta}z$ by the expression $2^{-1}(e^{i\alpha}wR_1^{-1} + e^{i\beta}zR_2^{-1})$.

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REFERENCES

- ¹ A. A. Temlyakov, *Dokl. Akad. Nauk SSSR*, **120**, No. 5 (1958).
- ² A. A. Temlyakov, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **21**, 89 (1957).
- ³ A. A. Temlyakov, *Uch. Zap. Mosk. Obl. Ped. Inst.*, **21**, 7 (1954).
- ⁴ G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Moscow-Leningrad, 1952.
- ⁵ I. I. Bavrin, *Dokl. Akad. Nauk SSSR*, **137**, No. 3 (1961).

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