
Soviet-era science, translated into English

MATHEMATICS

1961

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196101.75020>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

L. P. USOLTSEV

AN ANALOGUE OF THE FORTET-KAC THEOREM

(Presented by Academician I. M. Vinogradov, 1 XII 1960)

The following analogue of the Fortet-Kac theorem ⁽¹⁾ holds:

Theorem. Let $f(t)$ be a real-valued, bounded, periodic function with period 1, piecewise continuous, satisfying a Lipschitz condition in each interval of continuity. Suppose that

$$\int_0^1 f(t) dt = 0.$$

Let $g \geq 2$ be a fixed natural number; let p be a natural number such that $(g, p) = 1$; let $h = h(p)$ be a natural number such that $h \ll \log p / 2 \log g$ and $h \rightarrow \infty$ as $p \rightarrow \infty$. Let $N_p(\lambda)$ denote the number of integers a , $0 \leq a \leq p-1$, for which

$$\sum_{k=0}^{h-1} f\left(\frac{ag^k}{p}\right) < \lambda \sqrt{h}.$$

Then:

a) there exists the limit

$$\lim_{p \rightarrow \infty} \frac{1}{p} \sum_{a=0}^{p-1} \left(\frac{1}{\sqrt{h}} \sum_{k=0}^{h-1} f\left(\frac{ag^k}{p}\right) \right)^2 = \sigma^2;$$

b) if $\sigma \neq 0$,

$$\lim_{p \rightarrow \infty} \frac{N_p(\lambda)}{p} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\lambda} e^{-z^2/2\sigma^2} dz;$$

c) if $\sigma = 0$,

$$\lim_{p \rightarrow \infty} \frac{N_p(\lambda)}{p} = \begin{cases} 0 & \text{if } \lambda < 0, \\ 1 & \text{if } \lambda > 0. \end{cases}$$

The proof is carried out by the same method as in the work of A. G. Postnikov⁽²⁾; we apply the method of moments as was indicated in the work of M. P. Mineev⁽³⁾.

Introduce the following notation:

$$\|f\|^2 = \int_0^1 f^2(t) dt, \quad \|f\|_p^2 = \frac{1}{p} \sum_{a=0}^{p-1} f^2\left(\frac{a}{p}\right),$$

$$\rho(k) = \int_0^1 f(t)f(g^k t) dt, \quad \rho_p(k) = \frac{1}{p} \sum_{a=0}^{p-1} f\left(\frac{a}{p}\right) f\left(\frac{ag^k}{p}\right).$$

The following assertions are easily established:

$$\frac{1}{p} \sum_{a=0}^{p-1} f\left(\frac{ag^j}{p}\right) f\left(\frac{ag^k}{p}\right) = \rho_p(|k-j|);$$

as $p \rightarrow \infty$,

$$\frac{1}{p} \sum_{a=0}^{p-1} f^2\left(\frac{a}{p}\right) = \int_0^1 f^2(x) dx + O\left(\frac{1}{p}\right),$$

$$\frac{1}{p} \sum_{a=0}^{p-1} f\left(\frac{a}{p}\right) f\left(\frac{ag^k}{p}\right) = \int_0^1 f(x)f(g^k x) dx + O\left(\frac{g^k}{p}\right).$$

These assertions make it possible to prove that the limit exists

$$\lim_{p \rightarrow \infty} \frac{1}{p} \sum_{a=0}^{p-1} \left(\frac{1}{\sqrt{h}} \sum_{k=0}^{h-1} f\left(\frac{ag^k}{p}\right) \right)^2 = \sigma^2,$$

and from the proof it is clear that the quantity σ^2 coincides with the quantity introduced by Kac,

$$\sigma^2 = \lim_{p \rightarrow \infty} \frac{1}{p} \int_0^1 \left(\sum_{a=0}^{p-1} f(g^k t) \right)^2 dt.$$

Next, the following lemma will play the main role for us:

Lemma. *The second assertion of the theorem is valid for real-valued trigonometric polynomials.*

Proof. Let

$$\vartheta(t) = \sum_{m=-s}^s 'a_m e^{2\pi i m t}, \quad a_m = \overline{a_{-m}},$$

be one of such polynomials. Consider the expression

$$I_1 = \frac{1}{p} \sum_{a=0}^{p-1} \left(\sum_{x=0}^{h-1} \vartheta \left(\frac{ag^x}{p} \right) \right)^l,$$

where l is a fixed natural number. We have

$$\begin{aligned} I_1 &= \frac{1}{p} \sum_{a=0}^{p-1} \left(\sum_{x=0}^{h-1} \sum_{m=-s}^s a_m e^{2\pi i m \frac{ag^x}{p}} \right)^l = \\ &= \sum_{m_i=-s}^s 'a_{m_1} \cdots a_{m_l} \sum_{x_i=0}^{h-1} \frac{1}{p} \sum_{a=0}^{p-1} \exp \left[2\pi i \left(m_1 g^{x_1} + \cdots + m_l g^{x_l} \right) \frac{a}{p} \right] = \\ &= \sum_{m_i=-s}^s 'a_{m_1} \cdots a_{m_l} \sum_{x_i=0}^{h-1} 1; \quad i = 1, \dots, l; \quad m_1 g^{x_1} + \cdots + m_l g^{x_l} \equiv 0 \pmod{p}. \end{aligned}$$

Let us now count the number of solutions of the congruence

$$m_1 g^{x_1} + \cdots + m_l g^{x_l} \equiv 0 \pmod{p}$$

in the numbers $0 \leq x_1, \dots, x_l \leq h-1$. Consider the congruence

$$m_1 g^{x_1} + \cdots + m_k g^{x_k} \equiv n_1 g^{y_1} + \cdots + n_k g^{y_k} \pmod{p}, \quad (*)$$

where $m_1, \dots, m_k, n_1, \dots, n_k$ and $g \geq 2$ are fixed natural numbers;

$$0 \leq x_1, \dots, x_k, y_1, \dots, y_k \leq h-1; \quad h \leq \frac{\log p}{2 \log g}.$$

Take p so large that the strict inequality

$$|m_1| + \dots + |m_k| + |n_1| + \dots + |n_k| < \sqrt{p}$$

is satisfied. Then

$$|m_1 g^{x_1} + \dots + m_{kg}^{x_k} - n_1 g^{y_1} - \dots - n_{kg}^{y_k}| \leq$$

$$\leq (|m_1| + \dots + |m_k| + |n_1| + \dots + |n_k|) g^h < \sqrt{p} g^{\log p/2 \log g} = p,$$

therefore, for our p the number of solutions of the congruence $(*)$ in integers $0 \leq x_1, \dots, x_k, y_1, \dots, y_k \leq h-1$ is equal to the number of solutions of the equation $m_1 g^{x_1} + \dots + m_{kg}^{x_k} = n_1 g^{y_1} + \dots + n_{kg}^{y_k}$ in integers $0 \leq x_1, \dots, x_k, y_1, \dots, y_k \leq h-1$. The number of solutions of the latter equation was computed in the work of M. P. Mineev ⁽³⁾.

Following M. P. Mineev, we shall say that the compositions of the systems $m_1, \dots, m_k$ and $n_1, \dots, n_k$ coincide if one system is obtained from the other by some permutation of its elements. We shall call the system $\hat{m}_1, \dots, \hat{m}_k$ reduced with respect to g for the system $m_1, \dots, m_k$ if $\hat{m}_i = m_i / g^{l_i}$, $(\hat{m}_i, g) = 1$, $i = 1, \dots, k$. Then, suitably modifying the result of M. P. Mineev, we obtain the following assertion.

Let $\hat{m}_1, \dots, \hat{m}_k, \hat{n}_1, \dots, \hat{n}_k$ be the system reduced with respect to g for the system $m_1, \dots, m_k, n_1, \dots, n_k$; let $h \leq \log p/2 \log g$, and let p be sufficiently large, for example such that

$$|m_1| + \dots + |m_k| + |n_1| + \dots + |n_k| < \sqrt{p}.$$

Then:

- a) if the compositions of the systems $\hat{m}_1, \dots, \hat{m}_k$ and $\hat{n}_1, \dots, \hat{n}_k$ are distinct, then the number $A_k(h)$ of solutions of the congruence $(*)$ in the numbers $0 \leq x_1, \dots, x_k, y_1, \dots, y_k \leq h-1$ is estimated as $O(h^{k-1})$;
- b) if the compositions of the systems $\hat{m}_1, \dots, \hat{m}_k$ and $\hat{n}_1, \dots, \hat{n}_k$ coincide, then $A_k(h) = i_1! \dots i_s! h^k + O(h^{k-1})$, where $i_1, \dots, i_s$ are determined by the decomposition

$$\hat{m}_1^{(1)} = \hat{m}_2^{(1)} = \dots = \hat{m}_{i_1}^{(1)} = \hat{m}^{(1)}, \dots, \hat{m}_1^{(s)} = \dots = \hat{m}_{i_s}^{(s)} = \hat{m}^{(s)}, \quad \hat{m}^{(\alpha)} \neq \hat{m}^{(\beta)},$$

$$\text{if } \alpha \neq \beta, i_1 + i_2 + \dots + i_s = k.$$

Further, carrying out the calculations in the same way as was done in the work ⁽³⁾, we obtain

$$I_1 = \begin{cases} O(h^{(l-1)/2}), & \text{if } l \text{ is odd,} \\ C_l^{l/2} h^{l/2} (l/2)! \sigma^l / 2^{l/2} + O(h^{l/2-1}), & \text{if } l \text{ is even.} \end{cases}$$

Hence

$$\lim_{p \rightarrow \infty} \frac{1}{p} \sum_{a=0}^{p-1} \left(\frac{1}{\sqrt{h}} \sum_{x=0}^{h-1} \vartheta \left(\frac{ag^x}{p} \right) \right)^l = \begin{cases} 0, & \text{if } l \text{ is odd,} \\ C_l^{l/2} (l/2)! \sigma^l / 2^{l/2} = \sigma^l l! / 2^{l/2} (l/2)!, & \text{if } l \text{ is even.} \end{cases}$$

Consider

$$I_2 = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{\infty} z^l e^{-z^2/2\sigma^2} dz.$$

It is easy to verify that

$$I_2 = \begin{cases} 0, & \text{if } l \text{ is odd,} \\ \sigma^l l! / 2^{l/2} (l/2)!, & \text{if } l \text{ is even.} \end{cases}$$

Thus,

$$\lim_{p \rightarrow \infty} \frac{1}{p} \sum_{a=0}^{p-1} \left(\frac{1}{\sqrt{2h} \sigma} \sum_{x=0}^{h-1} \vartheta \left(\frac{ag^x}{p} \right) \right)^l = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^l e^{-t^2} dt.$$

Denote

$$k(a) = \frac{1}{\sqrt{2h} \sigma} \sum_{x=0}^{h-1} \vartheta \left(\frac{ag^x}{p} \right), \quad a = 0, 1, \dots, p-1.$$

Arrange

$k(a)$, $a = 0, 1, \dots, p-1$, in nondecreasing order $k(a_1) = \dots = k(a_{i_1}) < k(a_{i_1+1}) = \dots = k(a_{i_2}) < \dots < k(a_{i_{p-1}+1}) = \dots = k(a_{i_p})$, where $i_p = p$, and the system of numbers $a_1, \dots, a_p$ is obtained by some permutation of the system of numbers $0, 1, \dots, p-1$. We construct the following function, which will evidently be a distribution function:

$$\Phi_p(t) = \begin{cases} 0, & t < k(a_1) = \dots = k(a_{i_1}), \\ i_1/p, & k(a_1) = \dots = k(a_{i_1}) \leq t < k(a_{i_1+1}) = \dots = k(a_{i_2}), \\ \dots & \dots \\ i_{\rho-1}/p, & k(a_{i_{\rho-2}+1}) = \dots = k(a_{i_{\rho-1}}) \leq t < k(a_{i_{\rho-1}+1}) = \dots = k(a_{i_\rho}), \\ 1, & t \geq k(a_{i_\rho}) = k(a_p). \end{cases}$$

Let us compute the l -th moment of this function:

$$\begin{aligned} \int_{-\infty}^{\infty} t^l d\Phi_p(t) &= \frac{1}{p} (k^l(a_1) + \dots + k^l(a_{i_1})) + \dots + \frac{1}{p} (k^l(a_{i_{p-1}+1}) + \dots + k^l(a_{i_p})) = \\ &= \frac{1}{p} \sum_{a=0}^{p-1} \left(\frac{1}{\sqrt{2h}\sigma} \sum_{x=0}^{h-1} \vartheta \left(\frac{ag^x}{p} \right) \right)^l. \end{aligned}$$

Thus,

$$\lim_{p \rightarrow \infty} \int_{-\infty}^{\infty} t^l d\Phi_p(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^l e^{-t^2} dt.$$

By the second limit theorem of A. A. Markov ⁽⁴⁾,

$$\lim_{p \rightarrow \infty} \Phi_p(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^t e^{-u^2} du.$$

It is clear that $\Phi_p(t)$ is numerically equal to the number of expressions

$$\frac{1}{\sqrt{2h}\sigma} \sum_{x=0}^{h-1} \vartheta \left(\frac{ag^x}{p} \right),$$

less than or equal to t , divided by p , i.e.

$$\Phi_p(t) = N_p(\sqrt{2\sigma}t)/p.$$

Putting $u = z/\sqrt{2}\sigma$, we obtain

$$\lim_{p \rightarrow \infty} \frac{N_p(\sqrt{2\sigma}t)}{p} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\sqrt{2\sigma}t} e^{-z^2/2\sigma^2} dz;$$

putting $\lambda = \sqrt{2\sigma}t$, we finally obtain

$$\lim_{p \rightarrow \infty} \frac{N_p(\lambda)}{p} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\lambda} e^{-z^2/2\sigma^2} dz.$$

The lemma is proved.

The second assertion of the theorem follows from the lemma exactly as in Kac's theorem ⁽¹⁾. The last assertion of the theorem is proved in the same way as in Kac. The theorem is proved.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
30 XI 1960

REFERENCES

1. M. Kac, Ann. of Math., 2-d ser., **47**, No. 1, 33 (1946).
2. A. G. Postnikov, DAN, **133**, 1298 (1960).
3. M. P. Mineev, Izv. AN SSSR, ser. matem., **22**, 585 (1958).
4. A. A. Markov, *Calculus of Probabilities*, 4th ed., Moscow, 1924.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.