

Asymptotics of the Green's Function for Sobolev-Galpern Equations

B. R. Vainberg

1961

SovietRxiv

View the original and related papers at <https://sovietsrxiv.org/items/ru-196101.39861>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1961. Volume 136, No. 5

MATHEMATICS

B. R. VAINBERG

ASYMPTOTICS OF THE GREEN FUNCTION FOR SOBOLEV-GALPERN EQUATIONS

(Presented by Academician I. G. Petrovskii on 9 IX 1960)

§ 1. In the present work asymptotic formulas are obtained for the Green function (as $|x| \rightarrow \infty$ and fixed $t > 0$) for the Cauchy problem posed for Sobolev-Galpern equations with one spatial variable, i.e., equations of the form

$$P(\partial/\partial t, i\partial/\partial x)u = 0, \quad (1)$$

for which

$$\operatorname{Re} \lambda_j(\sigma) < c, \quad j = 1, 2, \dots, l, \quad (2)$$

where $P(\lambda, s)$ is a polynomial in two variables with constant coefficients; $s = \sigma + i\tau$; $\lambda_j(s)$ are the roots of the equation

$$P(\lambda, s) = P_l(s)\lambda^l + P_{l-1}(s)\lambda^{l-1} + \dots + P_0(s) \Big|_{t=0} = 0. \quad (3)$$

The Green function $G(x, t)$ is the solution of (1) with the initial conditions

$$G \Big|_{t=0} = \partial G / \partial t \Big|_{t=0} = \dots = \partial^{l-2} G / \partial t^{l-2} \Big|_{t=0} = 0, \quad \partial^{l-1} G / \partial t^{l-1} \Big|_{t=0} = \delta(x).$$

As a consequence of the asymptotic formulas, we obtain classes of existence for the solution of the Cauchy problem for equation (1)*.

To obtain the asymptotics of the Green function we use the saddle-point method. For equations correct in the sense of Petrovskii, the asymptotics of the Green function by the saddle-point method were obtained by M. V. Fedoryuk ⁽¹⁾. The Cauchy problem for equations not solved with respect to the highest derivative in t was first studied by S. L. Sobolev ⁽²⁾. The case of a general system of the indicated type with many independent variables was considered by S. A.

Galpern ^(3,4), who obtained a solution of the Cauchy problem and a uniqueness theorem in a certain class of functions. The question of existence and uniqueness for the case of exponentially growing functions was considered in the work of A. G. Kostyuchenko and G. I. Eskin ⁽⁵⁾. G. I. Eskin obtained, for equations (1) with one spatial variable, a more precise uniqueness theorem.

§ 2. Let us first consider an equation of first order in t ,

$$Q(i\partial/\partial x) \partial u/\partial t = P(i\partial/\partial x)u. \quad (4)$$

Expand $P(s)/Q(s)$ in a series in a neighborhood of infinity:

$$P(s)/Q(s) = \alpha_n s^n + \alpha_{n-1} s^{n-1} + \dots \quad (5)$$

By virtue of condition (2), only the following cases are possible:

- 1°. $\operatorname{Re} \alpha_n < 0$, n even, $n > 0$.
- 2°. $\operatorname{Re} \alpha_n = \operatorname{Re} \alpha_{n-1} = \dots = \operatorname{Re} \alpha_{p+1} = 0$, $\operatorname{Re} \alpha_p < 0$, $p > 0$, p even. (6)
- 3°. $\operatorname{Re} \alpha_n = \operatorname{Re} \alpha_{n-1} = \dots = \operatorname{Re} \alpha_1 = 0$; a) $n \geq 2$, b) $n < 2$.

Expand $P(s)/Q(s)$ in a neighborhood of the real root σ_k of the equation $Q(s) = 0$:

$$\frac{P(s)}{Q(s)} = \sum_{j=n}^0 \frac{\alpha_{jk}}{(s - \sigma_k)^j} + \frac{P_1(s)}{Q_1(s)}. \quad (7)$$

* By classes of uniqueness, existence, and correctness we shall understand them as defined by I. M. Gel'fand and A. I. Lidskiy.

Only the following cases are possible:

- 1⁰. $\operatorname{Re} \alpha_{n_k} < 0$, n_k even.
- 2⁰. $\operatorname{Re} \alpha_{n_k} = \dots = \operatorname{Re} \alpha_{(p+1)k} = 0$, $\operatorname{Re} \alpha_{p_k} < 0$, $p_k > 0$, p_k even.
- 3⁰. $\operatorname{Re} \alpha_{n_k} = \dots = \operatorname{Re} \alpha_{1_k} = 0$.

The asymptotics of the Green's function for equation (4) is determined by: 1) the behavior of $P(s)/Q(s)$ in a neighborhood of the real poles; 2) the behavior of $P(s)/Q(s)$ in a neighborhood of the nonreal poles; 3) the behavior of $P(s)/Q(s)$ as $|s| \rightarrow \infty$. Accordingly, it has three terms:

$$G(x, t) \sim G_1(x, t) + G_2(x, t) + G_3(x, t). \quad (9)$$

Denote:

$$1 + O\left(|x|^{-\frac{1}{n_k+1}}\right) = h, \quad 1 + O\left(|x|^{-\frac{1}{n-1}}\right) = g.$$

Theorem 1. A. For $G_1(x, t)$ in cases 1^0 (8), 2^0 (8), and 3^0 (8), respectively, the following formulas hold:

1^0 (8).

$$G_1(x, t) = \sum_{k=1}^m \left\{ e^{-r_k} \left\{ C_{1k} |x|^{-\frac{n_k+2}{2n_k+2}} t^{\frac{1}{2n_k+2}} h \exp \left[\gamma_{1k} |x|^{\frac{n_k}{n_k+1}} t^{\frac{1}{n_k+1}} h \right] \right. \right. \\ \left. \left. + C_{2k} |x|^{-\frac{n_k+2}{2n_k+2}} t^{\frac{1}{2n_k+2}} h \exp \left[\gamma_{2k} |x|^{\frac{n_k}{n_k+1}} t^{\frac{1}{n_k+1}} h \right] \right\} \right\}, \quad (10)$$

where $\text{Re } \gamma_{1k} < 0$, $\text{Re } \gamma_{2k} < 0$.

2^0 (8). If n_k is even, then as $|x| \rightarrow \infty$

$$G_1(x, t) = \sum_k \left\{ e^{-ix\sigma_k} \left\{ C_k |x|^{-\frac{n_k+2}{2n_k+2}} t^{\frac{1}{2n_k+2}} h \times \right. \right. \\ \left. \left. \times \exp \left[i\gamma_{1k} |x|^{\frac{n_k}{n_k+1}} t^{\frac{1}{n_k+1}} h + \gamma_{2k} |x|^{\frac{p_k}{n_k+1}} t^{\frac{n_k-p_k+1}{n_k+1}} h \right] \right\} \right\}; \quad (11)$$

here $\text{Re } \gamma_{2k} < 0$, γ_{1k} is real.

If n_k is odd, then $G_1(x, t)$ as $x \rightarrow \text{sign}(-i\alpha_{n_k})\infty$ will be the same as in case 1^0 (8), while as $x \rightarrow \text{sign}(i\alpha_{n_k})\infty$, $G_1(x, t)$ is the sum of two expressions of the form (11).

3^0 (8). If n_k is even, then as $|x| \rightarrow \infty$

$$G_1(x, t) = \sum \left\{ e^{-ix\sigma_k} \left\{ C_k |x|^{-\frac{n_k+2}{2n_k+2}} t^{\frac{1}{2n_k+2}} h \exp \left[i\gamma_k |x|^{\frac{n_k}{n_k+1}} t^{\frac{1}{n_k+1}} \right] \right\} \right\}; \quad (12)$$

here γ_k is real. If n_k is odd, then $G_1(x, t)$ as $x \rightarrow \text{sign}(-i\alpha_{n_k})\infty$ will be the same as in case 1^0 (8), while as $x \rightarrow \text{sign}(i\alpha_{n_k})\infty$, $G_1(x, t)$ is the sum of two expressions of the form (12).

B.

$$G_2(x, t) = \sum_{j=1}^n C_j |x|^{-\frac{n_j+2}{2n_j+2}} t^{\frac{1}{2n_j+2}} \exp \left\{ - \left[i x s_j \left[1 + O \left(|x|^{-\frac{1}{n_j+1}} \right) \right] \right] \right\}, \quad (13)$$

here $s_j = \sigma_j + i\tau_j$ is a complex root of the equation $Q(s) = 0$; η_j is its multiplicity, and only those roots for which $x\tau_j < 0$ are considered.

C. $G_3(x, t)$ in cases 1° (6), 2° (6), 3°a (6), and 3°b (6) has the form:

$$\begin{aligned} 1^\circ (6). \quad G_3(x, t) = & C_1 |x|^{-\frac{n-2}{2n-2}} t^{-\frac{1}{2n-2}} g \exp \left[\gamma_1 |x|^{\frac{n}{n-1}} t^{-\frac{1}{n-1}} \right] + \\ & + C_2 |x|^{-\frac{n-2}{2n-2}} t^{-\frac{1}{2n-2}} g \exp \left[\gamma_2 |x|^{\frac{n}{n-1}} t^{-\frac{1}{n-1}} g \right]; \end{aligned} \quad (14)$$

here $\operatorname{Re} \gamma_1 < 0$, $\operatorname{Re} \gamma_2 < 0$.

2° (6). If n is even, then as $|x| \rightarrow \infty$

$$G_3(x, t) = C |x|^{-\frac{n-2}{2n-2}} t^{-\frac{1}{2n-2}} g \exp \left[i \gamma_1 |x|^{\frac{n}{n-1}} t^{-\frac{1}{n-1}} g + \gamma_2 |x|^{\frac{p}{n-1}} t^{\frac{n-p+1}{n-1}} g \right]; \quad (15)$$

here γ_1 is real, $\operatorname{Re} \gamma_2 < 0$. If n is odd, then $G_3(x, t)$ as $x \rightarrow \operatorname{sign}(-i\alpha_n)\infty$ is the sum of two expressions of the form (15), while as $x \rightarrow \operatorname{sign}(i\alpha_n)\infty$, $G_3(x, t)$ will be the same as in case 1° (6).

3°a (6). If n is even, then as $|x| \rightarrow \infty$

$$G_3(x, t) = C |x|^{-\frac{n-2}{2n-2}} t^{-\frac{1}{2n-2}} g \exp \left[i \gamma |x|^{\frac{n}{n-1}} t^{-\frac{1}{n-1}} g \right]; \quad (16)$$

where γ is real. If n is odd, then $G_3(x, t)$ as $x \rightarrow \operatorname{sign}(-i\alpha_n)\infty$ is the sum of two expressions of the form (16), while as $x \rightarrow \operatorname{sign}(i\alpha_n)\infty$, $G_3(x, t)$ will be the same as in case 1° (6).

3°b (6). In this case $G_3(x, t) \equiv 0$.

§ 3. Let us now consider the arbitrary Sobolev–Galpern equation (1). We shall assume that for every s the roots $\lambda_j(s)$ are distinct. Let $\sigma_1, \dots, \sigma_m$ be real, and let $s_1 = \sigma_1 + i\tau_1, \dots, s_p = \sigma_p + i\tau_p$ be complex poles of $\lambda_j(s)$, with only those s_j for which $x\tau_j < 0$ being taken. We expand $\lambda_j(s)$ in a Newton–Puiseux series in a neighborhood of the infinitely remote point and in a neighborhood of the poles. Let, in a neighborhood of infinity,

$$\lambda_j(s) = \alpha_0 s^{k_0} + \alpha_1 s^{k_1} + \dots + \alpha_p s^{k_p} + \dots, \quad k_0 > k_1 > \dots,$$

and, in a neighborhood of the poles,

$$\lambda_j(s) = \alpha_0(s - \sigma_k)^{k_0} + \alpha_1(s - \sigma_k)^{k_1} + \dots + \alpha_p(s - \sigma_k)^{k_p} + \dots, \quad k_0 < k_1 < \dots.$$

In both cases $\alpha_0 \neq 0$, $k_i = l_i/m$, l_i and m are integers.

For a neighborhood of infinity, only the following cases are possible: 1) $k_0 \leq 0$; 2) $k_0 > k_1 > \dots > k_m > 0$, $k_{m+1} \leq 0$; $\operatorname{Re} \alpha_0 = \dots = \operatorname{Re} \alpha_m = 0$; then $k_0, k_1, \dots, k_m$ are integers; 3) $k_0 > k_1 > \dots > k_m > 0$, $k_{m+1} \leq 0$; $\operatorname{Re} \alpha_0 = \dots = \operatorname{Re} \alpha_{p-1} = 0$, $\operatorname{Re} \alpha_p \neq 0$, $p \leq m$; then $k_0, k_1, \dots, k_p$ are integers, p is even, and $\operatorname{Re} \alpha_p < 0$.

For the expansion in a neighborhood of a real pole σ_k , only the following cases are possible: 1) $k_0 < k_1 < \dots < k_m < 0$, $k_{m+1} \geq 0$; $\operatorname{Re} \alpha_0 = \dots = \operatorname{Re} \alpha_m = 0$; then $k_0, k_1, \dots, k_m$ are integers; 2) $k_0 < k_1 < \dots < k_m < 0$, $k_{m+1} \geq 0$; $\operatorname{Re} \alpha_0 = \dots = \operatorname{Re} \alpha_{p-1} = 0$, $\operatorname{Re} \alpha_p \neq 0$, $p \leq m$; then $k_0, k_1, \dots, k_p$ are integers, p is even, and $\operatorname{Re} \alpha_p < 0$.

These expansions make it possible to reduce the problem of finding the asymptotics of the Green's function for equation (1) to finding the asymptotics of the Green's function for equations (4) of first order in t . Let the Newton–Puiseux series expansions of the function

$$(-1)^n \Big/ \prod_{k, k \neq j} [\lambda_k(s) - \lambda_j(s)]$$

begin, in a neighborhood of infinity, with the term $a_j s^{\lambda_j}$, and in a neighborhood of the poles with the term $b_{jk} s^{\lambda_{jk}}$.

Theorem 2. Let $G(x, t)$ be the Green's function for the arbitrary equation (1). Then, as $|x| \rightarrow \infty$ and for constant $t > 0$,

$$G(x, t) \sim \sum_j G_{j_1}(x, t) + \sum_j G_{j_2}(x, t) + \sum_j G_{j_3}(x, t).$$

Here $G_{j_1}(x, t)$ depends only on the behavior of $\lambda_j(s)$ in a neighborhood of the real poles $\lambda_j(s)$; $G_{j_2}(x, t)$ depends only on the behavior of $\lambda_j(s)$ in a neighborhood of the complex poles; $G_{j_3}(x, t)$ depends only on the behavior of $\lambda_j(s)$ in a neighborhood of the infinitely remote point. $G_{j_1}(x, t)$ and $G_{j_2}(x, t)$ differ from the corresponding terms of the asymptotics of the Green's function for equation (4) by the factor

$$b_{jk} \left(\frac{|x|}{|\alpha_{n_k}|t} \right)^{-\frac{\lambda_{jk}}{n_k+1}},$$

and $G_{j_3}(x, t)$ —by the factor

$$a_j |x|^{\frac{\lambda_j}{n-1}} (|\alpha_n|t)^{-\frac{\lambda_j}{n-1}}.$$

Theorem 3. The asymptotics of the derivatives of the Green' s function for equation (1) is obtained by differentiating with respect to x and t the asymptotics of the Green' s function Γ .

§ 4. Let W be the class of existence of the solution of the Cauchy problem for equation (1). Each term $\sum_j G_{j_1}(x, t)$, $\sum_j G_{j_2}(x, t)$, and $\sum_j G_{j_3}(x, t)$ in the asymptotics of the Green' s function gives a class of functions W_i ($i = 1, 2, 3$), $W = \bigcap_i W_i$.

Let us describe the classes W_i .

$\varphi(x) \subset W_1$, if:

A^* . In certain neighborhoods of the real zeros of the polynomial $P_l(\sigma)$

$$\tilde{\varphi}(x) = |P_l(\sigma)|^{l-1} \psi(\sigma),$$

where $P_l(\sigma)$ is defined by formula (3), $\psi(\sigma)$ is a functional of the type of a function, and $\tilde{\varphi}(x)$ is the Fourier transform of the function $\varphi(x)$.

B. $\varphi(x)$ satisfies the following estimates:

1) If $\lambda(s)$ has a pole of type 1^0 (8), then

$$|\varphi(x)| < C_\varepsilon \exp \left[\varepsilon |x|^{\frac{n_k}{n_k+1}} \right]$$

for every $\varepsilon > 0$.

2) If $\lambda(s)$ has a pole of type 2^0 (8) with even n_k , then

$$|\varphi(x)| < C_\varepsilon \exp \left[\varepsilon |x|^{\frac{p_k}{n_k+1}} \right]$$

for every $\varepsilon > 0$.

3) If $\lambda(s)$ has a pole of type 2^0 (8) with odd n_k , then

$$|\varphi(x)| < C_\varepsilon \exp \left[\varepsilon |x|^{\frac{n_k}{n_k+1}} \right]$$

when $\text{sign}(ix\alpha_{n_k}) = -1$, for every $\varepsilon > 0$;

$$|\varphi(x)| < C_\varepsilon \exp \left[\varepsilon |x|^{\frac{p_k}{n_k+1}} \right]$$

when $\text{sign}(ix\alpha_{n_k}) = 1$, for every $\varepsilon > 0$.

4) If $\lambda(s)$ has a pole of type 3^0 (8) with even n_k , then

$$|\varphi(x)| < C_\varepsilon \left(|x|^{\frac{2\lambda_{jk}-2ln_k+n_k-\varepsilon}{2n_k+2}} + 1 \right)$$

for some $\varepsilon > 0$.

5) If $\lambda(s)$ has a pole of type 3^0 (8) with odd n_k , then

$$|\varphi(x)| < C_\varepsilon \exp \left[\varepsilon |x|^{\frac{n_k}{n_k+1}} \right]$$

when $\text{sign}(ix\alpha_{n_k}) = -1$, for every $\varepsilon > 0$;

$$|\varphi(x)| < C_\varepsilon \left(|x|^{\frac{2\lambda_{jk}-2ln_k+n_k-\varepsilon}{2n_k+2}} + 1 \right)$$

when $\text{sign}(ix\alpha_{n_k}) = 1$, for some $\varepsilon > 0$.

If $\lambda(s)$ has poles satisfying several of the conditions listed here at once, then the class W_1 is obtained as the intersection of the corresponding classes of functions.

$\varphi(x) \subset W_2$, if:

$$|\varphi(x)| < C_\varepsilon e^{(a_1-\varepsilon)|x|}$$

for $x < 0$, for some $\varepsilon > 0$;

$$|\varphi(x)| < C_\varepsilon e^{(a_2-\varepsilon)|x|}$$

for $x > 0$, for some $\varepsilon > 0$, where a_1 is the distance from the poles $\lambda(s)$ lying in the upper half-plane to the real axis, and a_2 is the distance from the poles lying in the lower half-plane to the real axis. The class W_3 is found in the same way as the classes of correctness in the work of M. V. Fedoryuk ⁽¹⁾.

If we take the intersection of the obtained classes of existence with the classes of uniqueness ^(3,4,6), then, since in our case the solution is the convolution of the initial data with the Green's function, we obtain the classes of correctness.

The author expresses gratitude to M. V. Fedoryuk for the assistance rendered in carrying out this work.

Moscow State University
named after M. V. Lomonosov

Received
8 IX 1960

CITED LITERATURE

1. M. V. Fedoryuk, DAN, **132**, No. 1 (1960).
2. S. L. Sobolev, Izv. AN SSSR, ser. matem., **18**, No. 1, 3 (1954).
3. S. A. Galpern, DAN, **119**, No. 4 (1957).
4. S. A. Galpern, Tr. Mosk. matem. obshch., **9**, p. 401 (1960).
5. A. G. Kostyuchenko, G. I. Eskin, UMN, **15**, issue 2, 211 (1960).
6. I. M. Gel'fand, G. E. Shilov, *Some Questions in the Theory of Differential Equations*, Moscow, 1958.

* An analogous condition was formulated in works ^(3,4), and later in work ⁽⁵⁾.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.