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Abstract

Full Text

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MATHEMATICS

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ON THE COMPLEXITY OF CIRCUITS IN CERTAIN BASES CONTAINING NONTRIVIAL ELEMENTS WITH ZERO WEIGHTS

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Let Boolean functions be realized by circuits ⁽¹⁾ of a certain type over a basis consisting of a set $\mathfrak{Z}$ of elements with zero weights, realizing a set of functions Z , and a set $\mathfrak{E}$ of elements with positive weights, realizing a set of functions E . We consider the cases when arbitrary circuits are allowed and when circuits are superpositions of basis elements ⁽²⁾.

As usual, we define the weight of a circuit and of a function in Shannon's sense $L(f), L(n)$.

1. Consider the basis $\mathfrak{A}$ with $E = \{\neg x\}$, $Z = \{x_1 \& x_2, x_1 \vee x_2, 0, 1\}$; the weight of negation is equal to 1.

We shall call a chain a set $\{\tilde{\sigma}^j\}$, $j = 0, 1, \dots, n$, of n -dimensional Boolean vectors such that $\tilde{\sigma}^{j'} < \tilde{\sigma}^{j'+1}$, $0 \leq j' \leq n-1$. Renumber all chains for fixed n ; denote the k -th chain by ω_k , $\omega_k = \{\tilde{\sigma}_k^0, \dots, \tilde{\sigma}_k^n\}$. Let $\xi(\tilde{x}) = \xi(x^1, \dots, x^n)$ be an arbitrary Boolean function. The vector $\tilde{\sigma}$ will be called a negative (positive) inversion node (n.i.n. or p.i.n.) of the pair (ξ, ω_k) , if $\tilde{\sigma} = \tilde{\sigma}_k^{j'}$ and $\xi(\tilde{\sigma}_k^{j'}) = 1$, $\xi(\tilde{\sigma}_k^{j'+1}) = 0$ ($\xi(\tilde{\sigma}_k^{j'}) = 0$, $\xi(\tilde{\sigma}_k^{j'+1}) = 1$). Denote by $M_k^-[\xi]$ ($M_k^+[\xi]$) the number of n.i.n. (p.i.n.) of the pair (ξ, ω_k) ; introduce the functions $M^-[\xi] = \max_k M_k^-[\xi]$,

$$M^+[\xi] = \max_k M_k^+[\xi].$$

We shall denote by $\chi(Q)$ the characteristic function of a set of vectors Q .

Lemma 1. 1) Every n.i.n. (p.i.n.) of one of the pairs $(\xi \& \eta, \omega_k)$, $(\xi \vee \eta, \omega_k)$ is an n.i.n. (p.i.n.) of one of the pairs (ξ, ω_k) , (η, ω_k) .

- 2) Every n.i.n. (p.i.n.) of the pair $(\neg \xi, \omega_k)$ is a p.i.n. (n.i.n.) of the pair (ξ, ω_k) .

Lemma 2. $M^+[\xi] - M^-[\xi] = \text{sign}[\xi(\tilde{1}) - \xi(\tilde{0})]$.

Theorem 1. For superpositions in the basis $\mathfrak{A}$,

$$L(f) = M^-[f].$$

Proof. By Lemmas 1 and 2, $M^-[f] \leq L(f)$. Denote by $R_k(\tilde{\sigma})$ the number of n.i.n. of the pair (f, ω_k) not less than the vector $\tilde{\sigma}$; the number $\max_k R_k(\tilde{\sigma})$ will be called the depth of the vector $\tilde{\sigma}$. Denote by Q_s the set of all vectors of depth s . We shall call a function η monotonically decreasing if $\eta(\tilde{\sigma}_1) \geq \eta(\tilde{\sigma}_2)$ for $\tilde{\sigma}_1 \leq \tilde{\sigma}_2$. Denote by ξ^+ (ξ^-) the least monotone (monotonically decreasing) function such that $\xi \leq \xi^+$. Put

$$g_s = (f \& \chi(\bigcup_{0 \leq s'' \leq s'} Q_{s''}))^+, \quad h_s = \neg(f \& \chi(Q_s))^-,$$

$1 \leq s \leq M^-[f]$, $0 \leq s' \leq M^-[f]$. We have

$$f = g_0 \vee \bigvee_{s=1}^{M^-[f]} g_s \& (\neg h_s),$$

where g_s, h_s are monotone functions. Consequently, $L(f) \leq M^-[f]$. The theorem is proved.

Corollary. $L(n) = \left\lfloor \frac{n}{2} \right\rfloor$.

If $f \neq \text{const}$, the theorem remains valid also for superpositions in bases obtained from $\mathfrak{A}$ by excluding one or both constants.

2. Let $\mathfrak{B}$ be a basis with $E = \{\neg x\}$, $Z = \{x_1 \vee x_2\}$; the weight of inversion is equal to 1.

Theorem 2. For superpositions in the basis $\mathfrak{B}$

$$L(n) \sim \frac{2^n}{n}.$$

The lower estimate is obtained as in (3). The upper estimate is based on representing a function in the form of a disjunction of conjunctions of two functions, one of which is symmetric and the other monotonically decreasing, with a subsequent proper representation (4) of the monotonically decreasing function.

Theorem 3. For circuits in the basis $\mathfrak{B}$

$$L(n) \sim \sqrt{2} 2^{n/2}.$$

The **proof** is based on the idea of using graphs, as in ⁽⁵⁾.

Theorem 4. For circuits in the basis Λ_i , $i = 1—5$ ⁽³⁾,

$$L(n) \sim 2^{n/2}.$$

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Note: Figure translations are in progress. See original paper for figures.

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