

On Projective Embeddings of Algebraic Varieties

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Abstract

Full Text

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ON PROJECTIVE EMBEDDINGS OF ALGEBRAIC VARIETIES

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MATHEMATICS

The well-known theorem of Kodaira ⁽¹⁾ asserts that if a divisor D on a compact Kähler manifold V is such that in the class of differential forms corresponding to the cohomology class dual to it there is found a form of type $(1, 1)$,

$$\sum h_{\alpha\beta} dz_{\alpha} \wedge d\bar{z}_{\beta}$$

such that the matrix $\|h_{\alpha\beta}\|$ is positive definite at every point of the manifold V , then the mapping of the manifold V into projective space corresponding to a sufficiently high multiple of the divisor D is a biregular embedding, i.e., it realizes the manifold V as a certain projective algebraic variety.

The question arises: what corresponds to this theorem in algebraic geometry?

Nakai ⁽²⁾ showed that, in order that a sufficiently high multiple of a divisor D on a nonsingular algebraic surface V_2 correspond to a biregular embedding of the surface V_2 in projective space, it is necessary and sufficient that the following conditions be satisfied:

$$D_2 > 0, \quad D \cdot C > 0,$$

where C runs through all one-dimensional subvarieties of the surface V_2 .

We shall assume that the ground field k is algebraically closed of characteristic zero. Let U_r be a nonsingular complete abstract algebraic variety of dimension r .

Theorem 1. Let D be a divisor on U_r such that $D^r > 0$ and $D^i \cdot C_i > 0$, where $i = 1, \dots, r-1$, and C_i runs through all subvarieties of the variety U_r having dimension i . Then, for sufficiently large n , the mapping of the variety U_r into projective space corresponding to the complete linear system $|nD|$ is a biregular embedding.

The following corollaries follow from Theorem 1:

Corollary 1. In order that a nonsingular complete abstract algebraic variety U_r be projective, it is necessary and sufficient that there exist on U_r a divisor D such that

$$D^r > 0, \quad D^i \cdot C_i > 0 \quad (i = 1, \dots, r-1)$$

for all i -dimensional subvarieties C_i of the variety U_r .

Corollary 2. Let D be a divisor on a nonsingular algebraic variety $U_r(\mathbb{C})$ over the field of complex numbers $\mathbb{C}$. Then, in order that in the class of differential forms corresponding to the dual—

to the divisor D in the cohomology class, there is found a form of type $(1, 1)$

$$\sum h_{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta$$

such that the matrix $\|h_{\alpha\beta}\|$ is positive definite at each point of the variety $U_r(C)$, it is necessary and sufficient that the conditions

$$D^r > 0, \quad D^i \cdot C_i > 0 \quad (i = 1, \dots, r-1)$$

hold for all i -dimensional subvarieties C_i of the variety $U_r(C)$.

Induction on the dimension is not directly applicable to the proof of Theorem 1. Therefore one has to prove a more general theorem, from which Theorem 1 is obtained as a special case:

Theorem 2. Let V_s be a complete abstract (possibly reducible) algebraic variety.

Suppose that there exists a regular mapping $f : V_s \rightarrow U_r$ such that to each point of the image fV_s there corresponds only a finite number of points on V_s . Suppose that a divisor C on U_r is such that $D^s \cdot fV_s > 0$, $D^i \cdot fC_i > 0$ ($i = 1, \dots, s-1$) for all i -dimensional subvarieties C_i of the variety V_s (where fC_i denotes the image of C_i under the mapping f). To the divisor nD there corresponds the fibred space of lines $\{nD\}$ over U_r , inducing a certain fibred space over V_s . Denote the latter by $\{nD\}_{V_s}$.

Then, for sufficiently large n , the mapping of the variety V_s into projective space, given by the group of sections of the fibred space $\{nD\}_{V_s}$, is a biregular embedding.

We give a brief outline of the proof of Theorem 2. The passage from $s-1$ to s is carried out in the following way. First the assertion of Theorem 2 is verified for normal V_s , then for irreducible ones, and after that for reducible ones.

An important role in the passage from normal varieties to irreducible ones is played by the following consideration. If V_s^* is the normalization of the irreducible variety V_s , then on V_s there exists such a positive divisor E that, if by E^* we denote its inverse image on V_s^* , by

$$\overline{O_{V_s^*}(-E^*)}$$

the sheaf over V_s^* whose stalk at a point $P \in V_s^*$ consists of functions F such that $(F) - E^* > 0$ locally at the point P , and by

$$\overline{O_{V_s^*}(-E^*)}$$

the direct image of the sheaf $O_{V_s^*}(-E^*)$ corresponding to the mapping $V_s^* \rightarrow V_s$, then there is an exact sequence

$$0 \rightarrow \overline{O_{V_s^*}(-E^*)} \rightarrow O_{V_s} \rightarrow K \rightarrow 0,$$

where O_{V_s} is the sheaf of local rings of the variety V_s , and K is a certain coherent sheaf over V_s , concentrated on the support of the divisor E . With the help of an analogous consideration one makes the passage from irreducible varieties to reducible ones.

After it has been proved that the mapping f_n , given by the group of sections of the fibred space $\{nD\}_{V_s}$, is regular and maps distinct points of V_s to distinct points of projective space, we argue as follows. Let $V_s^{(n)}$ be the image of V_s under the mapping f_n . $V_s^{(n)}$ is a projective algebraic variety. Obviously, f_n realizes a homeomorphism (in the Zariski topology) between V_s and $V_s^{(n)}$. Denote by $\bar{O}_{V_s}$ the image of the sheaf O_{V_s} over $V_s^{(n)}$. The sheaf obtained is coherent. Denote by $\bar{O}_{V_s}(m)$ the sheaf obtained from $\bar{O}_{V_s}$ by m -fold application of Serre's operation $F(m)$ (see (3)). It is not hard to see that the sheaf corresponding to the sheaf $\bar{O}_{V_s}(m)$ over V_s coincides with the sheaf

local sections of the fibred space $\{mnD\}_{V_s}$. Let $\alpha \in O_{V_s}(P)$ ($O_{V_s}(P)$ is the local ring of the point $P \in V_s$); y is a section of the fibred space $\{mnD\}_{V_s}$, not equal to zero at P ; $y_1, \dots, y_N$ is a basis of the group of sections of the fibred space $\{mnD\}_{V_s}$, and let $\bar{\alpha}, \bar{y}, \bar{y}_1, \dots, \bar{y}_N$ be the corresponding elements over $V_s^{(n)}$. By the well-known theorem of Serre (3), for sufficiently large m the coherent sheaf $F(m)$ over the projective algebraic variety V is generated by the elements of the group $H^0(V, F(m))$. Hence, if m is sufficiently large, then $\bar{\alpha}\bar{y} = \sum \beta_i \bar{y}_i$, where β_i belong to the local ring $O_{V_s^{(n)}}(f_{nP})$ (f_{nP} is the image of the point P on $V_s^{(n)}$). Hence

$$\bar{\alpha} = \sum \beta_i \frac{\bar{y}_i}{\bar{y}}.$$

Consider the mapping f_{mn} . There exists a regular mapping $t_{m,n}$ such that $t_{m,n}f_{mn} = f_n$:

$$\begin{array}{ccc}
 & V_s^{(mn)} & \\
 f_{mn} \swarrow & & \searrow t_{m,n} \\
 V_s & \xrightarrow{f_n} & V_s^{(n)}
 \end{array}$$

and, consequently, β_i belong to the local ring $O_{V_s^{(mn)}}(f_{mn}P)$.

Obviously, $\frac{\bar{y}_1}{\bar{y}}, \dots, \frac{\bar{y}_N}{\bar{y}}$ may be regarded as nonhomogeneous coordinates in a certain affine neighborhood of the point $f_{mn}P$. It follows that $\alpha \in O_{V_s^{(mn)}}(f_{mn}P)$, i.e. the sheaves $\bar{O}_{V_s}$ and $O_{V_s^{(mn)}}$ coincide. This proves that the mapping f_{mn} is biregular.

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