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Abstract

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MATHEMATICS

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ON AN INFINITE-DIMENSIONAL ANALOGUE OF THE LINEAR PROGRAMMING PROBLEM AND ITS APPLICATIONS TO SOME QUESTIONS IN APPROXIMATION THEORY

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In this note one of the possible infinite-dimensional analogues of the linear programming problem is formulated. For it an optimality criterion is established, which is then used to obtain a number of results in approximation theory.

1. Let C_E be the space of real-valued functions continuous on the compact set E ; let V_E be the space of functions of bounded variation defined on the system of Borel subsets of E . Consider the problem consisting in determining a function $x \in V_{E_1}$ minimizing the Radon-Stieltjes integral

$$\int_{E_1} c(\tau) dx \quad (1)$$

under the conditions:

$$\int_{\tau \in E_1} a_1(t, \tau) dx \geq b_1(t), \quad t \in E_2, \quad (2)$$

$$\int_{\tau \in E_1} a_2(t, \tau) dx = b_2(t), \quad t \in E_3. \quad (3)$$

Here E_i , $i = 1, 2, 3$, are arbitrary compact sets; $c(\tau) \in C_{E_1}$, $b_i(t) \in C_{E_{i+1}}$, $a_i(t, \tau) \in C_{E_1 \times E_{i+1}}$, $i = 1, 2$. Problem (1)–(3) is a generalization to the infinite-dimensional case of the general linear programming problem. A function $x \in V_{E_1}$ satisfying conditions (2), (3) will be called **feasible**.

A feasible function delivering the conditional minimum for (1) is called **optimal**. We shall say that problem (1)–(3) is **regular** if there is an $\varepsilon > 0$ such that for every function

$$b(t) \in C_{E_3}, \quad |b(t) - b_2(t)|_{C_{E_3}} \leq \varepsilon,$$

there exists a function $\bar{x} \in V_{E_1}$ satisfying conditions (3), where $b_2(t)$ is replaced by $b(t)$, and turning (2) into strict inequalities.

Theorem 1 (optimality criterion for problem (1)–(3)). *For the optimality of a feasible function x^* , it is sufficient, and in the case of regularity of problem (1)–(3) also necessary, that there exist a nonnegative function $y_1 \in V_{E_2^*}$ and a function $y_2 \in V_{E_3}$ such that*

$$c(\tau) = \int_{t \in E_2^*} a_1(t, \tau) dy_1 + \int_{t \in E_3} a_2(t, \tau) dy_2.$$

Here E_2^* denotes the set of points $t \in E_2$ for which

$$\int_{\tau \in E_1} a_1(t, \tau) dx^* = b_1(t).$$

We note that the functions y_1 and y_2 are analogues of the resolving multipliers of L. V. Kantorovich ⁽¹⁾.

Important for applications is the following special case of problem (1)–(3).

Determine the vector $X = (x_1, x_2, \dots, x_N)$ minimizing

$$\sum_{j=1}^N c_j x_j \tag{4}$$

under the conditions

$$\sum_{j=1}^N a_j(t) x_j \geq b(t), \quad t \in E; \tag{5}$$

$$\sum_{j=1}^N a_{ij} x_j = b_i, \quad i = 1, 2, \dots, r. \tag{6}$$

Here E is an arbitrary compact set; the vectors $(a_{i1}, a_{i2}, \dots, a_{iN})$, $i = 1, 2, \dots, r$, are linearly independent; $a_j(t), b(t) \in C_E$.

We shall assume the existence of a feasible vector of problem (4)–(6) that turns (5) into strict inequalities (the regularity condition for the problem). Using the general optimality criterion (Theorem 1), one can establish the following assertion:

Theorem 2 (optimality criterion for problem (4)–(6)). *For the optimality of a feasible vector X^* it is necessary and sufficient that there exist points $t_i \in E$,*

$$\sum_{j=1}^N a_j(t_i) x_j^* = b(t_i),$$

and numbers $\lambda_i > 0$, $i = 1, 2, \dots, s$; $\mu_1, \mu_2, \dots, \mu_r$, such that:

a)

$$\sum_{i=1}^s \lambda_i a_j(t_i) + \sum_{i=1}^r \mu_i a_{ij} = c_j, \quad j = 1, 2, \dots, N;$$

b) the vectors $(a_1(t_i), a_2(t_i), \dots, a_N(t_i))$, $i = 1, 2, \dots, s$, and $(a_{i1}, a_{i2}, \dots, a_{iN})$, $i = 1, 2, \dots, r$, are linearly independent.

2. Let us consider the following generalization of Chebyshev's problem of best approximation.

Let $f(\tau)$ and $\varphi_j(\tau)$, $j = 1, 2, \dots, n$, be arbitrary functions continuous on the compact set K , with the $\varphi_j(\tau)$ assumed linearly independent. It is required to find a generalized polynomial

$$P(\tau) = \sum_{j=1}^n d_j \varphi_j(\tau),$$

deviating least from $f(\tau)$ in the metric C_K under the conditions:

$$\sum_{j=1}^n d_j a_j(t) \geq b(t), \quad t \in E; \quad (7)$$

$$\sum_{j=1}^n d_j a_{ij} = b_i, \quad i = 1, 2, \dots, r. \quad (8)$$

We shall assume that the assumptions stated above concerning the constraints (5), (6) are satisfied for (7), (8). Then the problem is solvable.

For the formulated approximation problem there holds an analogue of P. L. Chebyshev's theorem, which is a consequence of Theorem 2.

Theorem 3. In order that the polynomial

$$P^*(\tau) = \sum_{j=1}^n d_j^* \varphi_j(\tau),$$

satisfying conditions (7), (8), be a polynomial of best approximation to the function $f(\tau)$, it is necessary and sufficient that there exist points

$$\tau_i \in K, \quad \max_{\tau \in K} |f(\tau) - P^*(\tau)| = |f(\tau_i) - P^*(\tau_i)|, \quad i = 1, 2, \dots, k; \quad k \geq 1;$$

$$t_i \in E, \quad \sum_{j=1}^n a_j(t_i) d_j^* = b(t_i), \quad i = 1, 2, \dots, s,$$

such that:

a) the matrix

$$M = \begin{pmatrix} \varphi_1(\tau_1) & \varphi_2(\tau_1) & \dots & \varphi_n(\tau_1) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \varphi_1(\tau_k) & \varphi_2(\tau_k) & \dots & \varphi_n(\tau_k) \\ a_1(t_1) & a_2(t_1) & \dots & a_n(t_1) \\ \dots & \dots & \dots & \dots \\ a_1(t_s) & a_2(t_s) & \dots & a_n(t_s) \\ a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{r1} & a_{r2} & \dots & a_{rn} \end{pmatrix}$$

has rank equal to $p - 1$, where $1 \leq p = k + s + r \leq n + 1$;

- b) in the matrix M there are $p - 1$ columns such that, for any i , $1 \leq i \leq p$, the determinant Δ_i , formed from the elements lying at the intersection of these columns and the rows of M , with the i -th row omitted, is nonzero;
- c) $(-1)^{i+\nu} \text{sign } \Delta_i = \text{sign}[f(\tau_i) - P^*(\tau_i)]$, $i = 1, 2, \dots, k$; $(-1)^{i+\nu} \text{sign } \Delta_i > 0$, $i = k + 1, \dots, k + s$; $\nu = 0$ or 1 .

The approximation problem considered is a generalization both of the ordinary problem of Chebyshev approximation and of the problems of V. A. Markov ⁽²⁾ and A. P. Psheborky ⁽³⁾, in which only constraints (8) appeared. Therefore Theorem 3 contains, in particular, the necessary and sufficient conditions that the polynomial of best approximation must satisfy as applied to each of these problems (see ^(4, 5, 6)).

For the generalized problem of P. L. Chebyshev the following uniqueness theorem holds, which is an analogue of the well-known theorem of A. Haar.

Theorem 4. In order that the generalized problem of P. L. Chebyshev under constraints (7), (8) have a unique solution for any functions $f(\tau)$, $b(t)^*$, continuous respectively on K and E , it is necessary and sufficient that there not exist systems of points

$$\tau_i \in K, \quad i = 1, 2, \dots, k; \quad k \geq 1;$$

$$t_i \in E, \quad i = 1, 2, \dots, s,$$

satisfying the conditions:

a)

$$\sum_{i=1}^k \alpha_i \varphi_j(\tau_i) + \sum_{i=1}^s \beta_i a_j(t_i) + \sum_{i=1}^r \gamma_i a_{ij} = 0, \quad j = 1, 2, \dots, n;$$

b) none of the numbers α_i, β_i is equal to zero;

c) $\text{sign } \beta_1 = \dots = \text{sign } \beta_s, \quad k + s + r \leq n$.

The proof of Theorem 4 rests on the assertion of Theorem 3. Theorems 3 and 4 can be refined under various assumptions concerning the functions $\varphi_i(\tau)$ and conditions (7), (8).

* Let us recall that all the problems under consideration are assumed to be regular, so that we are speaking only of those functions $b(t)$ for which the regularity conditions turn out to be fulfilled.

We note that, on the basis of the methods of linear programming, one can indicate a number of numerical algorithms for solving the generalized P. L. Chebyshev problem presented here.

3. The optimality criteria for problems (1)–(3), (4)–(6) can be used effectively to obtain a number of exact estimates in approximation theory. As an example, let us consider the following problem.

Suppose that for every k , $1 \leq k \leq n$, the real-valued functions $\varphi_j(\tau)$, $j = 1, 2, \dots, k$, form a P. L. Chebyshev system on $[a, b]$, $\varphi_1(\tau) = 1$.

By R_n denote the set of generalized polynomials

$$P(\tau) = \sum_{j=1}^n d_j \varphi_j(\tau),$$

bounded on $[a, b]$ in absolute value by 1. It is required to indicate necessary and sufficient conditions on the numbers γ_j , $j = 1, 2, \dots, n$, in order that

$$\sup_{P \in R_n} \left| \sum_{j=1}^n \gamma_j d_j \right|$$

be attained at the polynomial

$$P^*(\tau) = \sum_{j=1}^n d_j^* \varphi_j(\tau) \neq \text{const.}$$

The solution of this problem is given by the following assertion, which is a consequence of Theorem 2.

Theorem 5. *Let $\tau_1, \tau_2, \dots, \tau_p$ be all those points of $[a, b]$ at which $|P^*(\tau)| = 1$ (obviously, $p \leq n$); let the numbers $\delta_1, \delta_2, \dots, \delta_p$ be determined by the system of equations*

$$\sum_{i=1}^p \delta_i \varphi_j(\tau_i) = \gamma_j, \quad j = 1, 2, \dots, p.$$

The desired necessary and sufficient conditions have the form:

$$\text{a) } (-1)^\nu \delta_j P^*(\tau_j) \geq 0, \quad j = 1, 2, \dots, p; \quad \nu = 0 \text{ or } 1;$$

$$\text{b) } \gamma_j = \sum_{i=1}^p \delta_i \varphi_j(\tau_i) \quad \text{for } j = p+1, \dots, n.$$

For

$$\varphi_j(\tau) = \begin{cases} \cos(j-1)\tau, & j = 1, 2, \dots, s+1, \\ \sin(j-s-1)\tau, & j = s+2, \dots, 2s+1 = n; \end{cases} \quad 0 \leq \tau \leq 2\pi \quad (9)$$

and $P^*(\tau) = \cos(s\tau - \alpha)$, where $\alpha = \gamma_{s+1} / \sqrt{\gamma_{s+1}^2 + \gamma_{2s+1}^2}$, the problem was solved by S. N. Bernstein⁽⁷⁾ (see also⁽⁸⁾). S. N. Bernstein's result follows easily from condition a) of Theorem 5. For $\varphi_j(\tau)$ of the form (9) and arbitrary $P^*(\tau)$, the problem was considered by Rogosinski⁽⁹⁾. The main results of⁽⁹⁾ are contained in Theorem 5.

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