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Abstract

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MATHEMATICS

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ON THE CONVEXITY OF THE DOMAIN OF REGULARITY OF A SOLUTION OF A DIFFERENTIAL EQUATION OF INFINITE ORDER

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Let

$$Dy = y^{(s)} + p_1(z)y^{(s-1)} + \dots + p_s(z)y, \quad (1)$$

where $p_1(z), \dots, p_s(z)$ are entire functions, and let

$$D^0 y = y(z), \quad D^k y = D(D^{k-1}y).$$

Introduce the equation

$$M(y) \equiv \sum_0^\infty c_n D^n y = F(z). \quad (2)$$

Suppose that $F(z)$ is an entire function and that the constants c_m are such that the characteristic function

$$L(z) = \sum_0^\infty c_m z^m$$

belongs to the class $[1/s, 0]$, i.e. for every $\varepsilon > 0$ it satisfies the condition

$$|L(z)| < e^{\varepsilon|z|^{1/s}}, \quad |z| > r_0(\varepsilon).$$

It can be shown (see, for example, ⁽¹⁾, p. 205) that the series standing on the left-hand side of (2) converges at every point at which the function $y(z)$ is regular, and its sum is regular at that point.

Theorem. *The domain of existence of any solution of equation (2) is convex.*

For $Dy = y'$ we obtain from this Polya's theorem (2).

The proof of the theorem is based on the following lemma, which is also of independent interest. Denote by $y(z, \lambda)$ the solution of the equation

$$Dy = \lambda^s y$$

(here λ is a parameter) satisfying the initial conditions

$$y(z_0, \lambda) = 1, \quad y'(z_0, \lambda) = \lambda, \dots, y^{(s-1)}(z_0, \lambda) = \lambda^{s-1}.$$

Lemma. If

$$\gamma(z) = \sum_0^\infty \frac{\alpha_m}{(z - z_0)^{m+1}}, \quad \overline{\lim}_{m \rightarrow \infty} \sqrt[m]{|\alpha_m|} = \rho < \infty,$$

then the function

$$F(\lambda) = \frac{1}{2\pi i} \int_{|z - z_0| = \rho_1 > \rho} \gamma(z) y(z, \lambda) dz$$

is an entire function of order one of type ρ .

For the proof of the lemma, put

$$y(z, \lambda) = \sum_0^\infty A_m(z) \lambda^m. \quad (3)$$

Here $A_m(z)$ are entire functions. In terms of them one may expand any function analytic in a neighborhood of the point z_0 . Namely, M. K. Fage proved³ that if $f(z)$ is regular in the disk $|z - z_0| < r$, then in some neighborhood of the point z_0 the following representation holds:

$$f(z) = \sum_{m=0}^\infty \left\{ \frac{d^q}{dz^q} D^p f \right\}_{z=z_0} A_m(z), \quad m = ps + q, \quad 0 \leq q < s. \quad (4)$$

If one takes into account M. K. Fage's estimate for $|A_m(z)|$,

$$|A_m(z)| < C \frac{|z - z_0|^m}{m!} \quad (5)$$

(C is a constant, in general different for each disk), and the estimate for $|(D^p f)^{(q)}|$,

$$|(D^p f)^{(q)}_{z=z_0}| < \frac{m!}{r^m} \frac{M(R, f)}{(1 - r/R)^{NR+1}}, \quad 0 < r < R, \quad (6)$$

$$N = \max_{1 \leq j \leq s} [1, M(R; p_j)], \quad M(R, \varphi) = \max_{|z-z_0|=R} |\varphi(z)|$$

(in the case $s = 2$ it was obtained in paper⁴; in the general case it is proved analogously), then we obtain that the series (4) converges, in any event, in the disk $|z - z_0| < \rho$, whose radius ρ is equal to the distance from z_0 to the nearest singular point of $f(z)$.

Put

$$\frac{1}{2\pi i} \int_{|z-z_0|=\rho_1} \gamma(z) A_m(z) dz = \frac{\beta_m}{m!} \quad (7)$$

and show that

$$\overline{\lim}_{m \rightarrow \infty} \sqrt[m]{|\beta_m|} = \rho. \quad (8)$$

Suppose the contrary, i.e., that

$$\overline{\lim}_{m \rightarrow \infty} \sqrt[m]{|\beta_m|} = \mu < \rho \quad (9)$$

(this limit, by virtue of (5), cannot be greater than ρ). Consider the function

$$\Phi(\lambda) = \frac{1}{2\pi i} \int_{|z-z_0|=\rho_1} \gamma(z) e^{\lambda(z-z_0)} dz.$$

This function is entire of order one and type ρ . On the basis of (4) we have

$$e^{\lambda(z-z_0)} = \sum_{m=0}^{\infty} [D^p e^{\lambda(z-z_0)}]_{z=z_0}^{(q)} A_m(z).$$

Consequently, taking (7) into account, we obtain

$$\Phi(\lambda) = \sum_{m=0}^{\infty} \frac{\beta_m}{m!} [D^p e^{\lambda(z-z_0)}]_{z=z_0}^{(q)}.$$

Let us estimate the right-hand side in modulus. To do this we use estimate (6), putting in it $f(z) = e^{\lambda(z-z_0)}$, $r = \mu + 2\varepsilon$, $R = \mu + 3\varepsilon$, and the condition (9), by virtue of which

$$|\beta_m| < (\mu + \varepsilon)^m, \quad m > m_0(\varepsilon).$$

We shall have

$$|\Phi(\lambda)| < \frac{e^{|\lambda|(\mu+3\varepsilon)}}{\left(1 - \frac{\mu+2\varepsilon}{\mu+3\varepsilon}\right)^{NR+1}} \left\{ \sum_{m=0}^{m_0} \frac{|\beta_m|}{(\mu+2\varepsilon)^m} + \sum_{m_0+1}^{\infty} \left(\frac{\mu+\varepsilon}{\mu+2\varepsilon}\right)^m \right\} < e^{(\mu+4\varepsilon)|\lambda|},$$

$$|\lambda| > \lambda_0.$$

For small $\varepsilon > 0$ the number $\mu + 4\varepsilon < \rho$, and therefore $\Phi(\lambda)$ is an entire function of the first order of type less than ρ , which is impossible. Consequently, equality (8) is valid. By virtue of this equality, taking (3) into account, we obtain that the function

$$F(\lambda) = \frac{1}{2\pi i} \int_{|z-z_0|=\rho_1} \gamma(z)y(z, \lambda) dz = \sum_0^{\infty} \frac{\beta_m}{m!} \lambda^m$$

is indeed an entire function of the first order of type ρ .

For what follows, along with Dy and $M(y)$, we introduce the operators

$$\widetilde{D}y = y^{(s)} + [p_1(z)y]^{(s-1)} + \dots + p_s(z)y, \quad \widetilde{M}y = \sum_0^{\infty} c_m \widetilde{D}^m y$$

and denote by $\tilde{y}(z, \lambda)$ the solution of the equation $\widetilde{D}y = \lambda^s y$, satisfying at the point z_0 the same conditions as $y(z, \lambda)$.

Proof of the theorem. Let $y(z)$ be a solution of equation (2). Suppose the contrary, i.e., that the domain G of existence of the function $y(z)$ is not convex. It will follow from this that through some boundary point ξ of the domain G one can draw such a circle C that some arc (α, β) of it has with the boundary of the domain G only one common point $\xi \neq \alpha, \beta$ and is turned with its convexity into the domain G (a proof of this fact is given in paper ⁽⁵⁾, p. 72; the corresponding figure is also given there). Join the points α and β in the domain G by two paths L_1 and L_2 . Let, moreover, L_2 lie in the domain bounded by the curve L_1 and the arc (α, β) of the circle C . When the point z lies in the domain E bounded by the curves L_1 and L_2 , by Cauchy's formula we have

$$y(z) = \frac{1}{2\pi i} \int_{L_1} \frac{y(t) dt}{t-z} + \frac{1}{2\pi i} \int_{L_2} \frac{y(t) dt}{t-z} = y_1(z) + y_2(z).$$

The function $y_1(z)$ is regular everywhere outside L_1 ; in particular, it is regular at the point ξ . Using the equality $y_2(z) = y(z) - y_1(z)$, we shall continue the function $y_2(z)$ from the domain E through the curve L_2 . We see that $y_2(z)$ is regular everywhere outside the circle C , while the point ξ is a singular point for it. Note also that $y_2(\infty) = 0$. Draw through the point ξ a circle C_1 , tangent to C at the point ξ and enclosing C . The function $y_2(z)$ is regular outside C_1 and at all points of C_1 different from ξ . We shall show that the point ξ is singular for the function

$$M(y_2) = \sum_0^{\infty} c_m D^m y_2(z).$$

This function is regular outside C_1 and at all points of C_1 distinct from ξ . Let z_0 and ρ be, respectively, the center and the radius of the circle C_1 . Outside C_1

$$y_2(z) = \sum_0^{\infty} \frac{a_m}{(z - z_0)^{m+1}}, \quad \lim_{m \rightarrow \infty} \sqrt[m]{|a_m|} = \rho.$$

We have

$$\frac{1}{2\pi i} \int_{|z-z_0|=\rho_1 > \rho} D^m [y_2(z)] \tilde{y}(z, \lambda) dz = \frac{1}{2\pi i} \int_{|z-z_0|=\rho_1} y_2(z) \tilde{D}^m [\tilde{y}(z, \lambda)] dz,$$

whence

$$\omega(\lambda) = \frac{1}{2\pi i} \int_{|z-z_0|=\rho_1} M[y_2(z)] \tilde{y}(z, \lambda) dz = \frac{1}{2\pi i} \int_{|z-z_0|=\rho_1} y_2(z) \tilde{M}[\tilde{y}(z, \lambda)] dz.$$

Since

$$\tilde{M}[\tilde{y}(z, \lambda)] = L(\lambda^s) \tilde{y}(z, \lambda),$$

it follows that

$$\omega(\lambda) = L(\lambda^s) \omega_1(\lambda), \quad \omega_1(\lambda) = \frac{1}{2\pi i} \int_{|z-z_0|=\rho_1} y_2(z) \tilde{y}(z, \lambda) dz.$$

By the lemma, $\omega_1(\lambda)$ is an entire function of order one of type ρ . By the hypothesis, $L(\lambda^s)$ is an entire function of class $[1, 0]$. Therefore $\omega(\lambda)$ is an entire function of order one of type ρ . From this, on the basis of the lemma, it follows that the function $M(y_2)$ has at least one singular point on the circle C_1 . This point is the point ξ . It is now easy to arrive at a contradiction. Equation (2) gives

$$M(y_1) + M(y_2) = F(z).$$

The first term is regular at the point ξ , while the second term has a singularity at the point ξ ; consequently, the point ξ is singular for $F(z)$, which is impossible. Hence the domain G is convex.

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Note: Figure translations are in progress. See original paper for figures.

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