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Abstract

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MATHEMATICS

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THE CAUCHY TRANSFORM OF GENERALIZED FUNCTIONS

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1°. Let the simple smooth closed contour L_k divide the plane of the complex variable z_k ($k = 1, 2, \dots, n$) into two parts: the interior D_k^+ and the exterior D_k^- . Put:

$$\Delta_{p_1 \dots p_m}^+ = D_1^+ \times \dots \times D_{p_1-1}^+ \times D_{p_1}^- \times D_{p_1+1}^+ \times \dots \times D_{p_m-1}^+ \times D_{p_m}^- \times D_{p_m+1}^+ \times \dots \times D_n^+,$$

$$\tau_{p_1 \dots p_m} = (\tau_1, \dots, \tau_{p_1-1}, t_{p_1}, \tau_{p_1+1}, \dots, \tau_{p_m-1}, t_{p_m}, \tau_{p_m+1}, \dots, \tau_n),$$

$$t_{p_1 \dots p_m} = (t_1, \dots, t_{p_1-1}, t_{p_1+1}, \dots, t_{p_m-1}, t_{p_m+1}, \dots, t_n), \quad \text{where } t_j \in L_j, \tau_j \in L_j,$$

$$d\tau_{p_1 \dots p_m} = d\tau_1 \dots d\tau_{p_1-1} d\tau_{p_1+1} \dots d\tau_{p_m-1} d\tau_{p_m+1} \dots d\tau_n$$

$$(0 \leq m \leq n, \quad 1 \leq p_1 < \dots < p_m \leq n),$$

in particular: $t = (t_1, \dots, t_n)$, $\tau = (\tau_1, \dots, \tau_n)$, $d\tau = d\tau_1 \dots d\tau_n$, and $z = (z_1, \dots, z_n)$;
 $L = L_1 \times \dots \times L_n$; $\{1, 2, \dots, n\} \setminus \{p_1, p_2, \dots, p_m\} = \{q_1, q_2, \dots, q_{n-m}\}$, $1 \leq m \leq n$;

$$\varphi_j = S_j \varphi = \frac{1}{\pi i} \int_{L_j} \frac{\varphi(t\tau_j) d\tau_j}{\tau_j - t_j}, \quad 1 \leq j \leq n, \quad (1)$$

$$\varphi_{p_1 \dots p_m} = S_{p_1 \dots p_m} \varphi = S_{p_1} \dots S_{p_m} \varphi, \quad \varphi_{\overline{p_1 \dots p_m}} = \varphi_{q_1 \dots q_{n-m}}, \quad \tilde{\varphi} = S\varphi = S_{1 \dots n} \varphi$$

and

$$S_0\varphi = \varphi.$$

If the function $\varphi(t)$ on L satisfies a Hölder condition with exponents $\alpha_1, \dots, \alpha_n$, then we shall write $\varphi(t) \in H(\alpha)$; if, moreover, $\varphi(t)$ is differentiable k_j times with respect to t_j , and

$$D_{k_1 \dots k_n} \varphi(t) = \frac{\partial^{k_1 + \dots + k_n} \varphi(t)}{\partial t_1^{k_1} \dots \partial t_n^{k_n}} \in H(\alpha),$$

then we shall write $\varphi(t) \in H_k(\alpha)$. If $\varphi(t) \in H(\alpha)$, then the limiting values, as $z \rightarrow t \in L$ and $z \in \Delta_{p_1 \dots p_m}^-$, of the Cauchy-type integral

$$\Phi(z) = \frac{1}{(2\pi i)^n} \int_L \frac{\varphi(\tau) d\tau}{\prod_{k=1}^n (\tau_k - t_k)}, \quad (2)$$

which we shall denote by $\Phi_{p_1 \dots p_m}^-(t)$, exist ⁽¹⁾ and are computed by the Sokhotski formulas

$$2^n \Phi_{p_1 \dots p_m}^-(t) = \left\{ \prod_{\mu=1}^m (S_0 + S_{p_\mu}) \prod_{\nu=1}^{n-m} (-S_0 + S_{q_\nu}) \right\} \varphi,$$

where the singular operators (1) are multiplied according to the rules:

$$S_0 S_j = S_j, \quad S_k S_j = S_j S_k = S_{jk}, \quad S_j^2 = S_0, \quad j = 1, 2, \dots, n.$$

2°. Theorem 1. Let $\varphi(t) \in H(\alpha)$; then the values of the integral (2) near L satisfy the condition

$$|\Phi(z) - \Phi_{p_1 \dots p_m}^-(t)| \leq M \sum_{k=1}^n |t_k - z_k|^{\beta_k}, \quad M > 0, \quad 0 < \beta_k < \alpha_k, \quad k = 1, 2, \dots, n.$$

Theorem 1 is proved in the same way as the analogous theorem in the case of one variable ⁽³⁾.

Lemma*. Let $\varphi(t) \in H_k(\alpha)$; then

$$\lim_{z \rightarrow t} D_{l_1 \dots l_n} \Phi(z) = [D_{l_1 \dots l_n} \Phi(t)]_{p_1 \dots p_m}^- = D_{l_1 \dots l_n} \Phi_{p_1 \dots p_m}^-(t),$$

where $z \in \Delta_{p_1 \dots p_m}^-$, $t \in L$, $0 \leq m \leq n$, $1 \leq p_1 < \dots < p_m \leq n$, $0 \leq l_j \leq k_j$, $j = 1, 2, \dots, n$.

The lemma is proved by mathematical induction, using the fundamental Cauchy theorem, the Sokhotski formulas, and Theorem 1.

Theorem 2. If $\varphi(t) \in H_k(\alpha)$, then the equalities

$$D_{l_1 \dots l_n} S_{r_1 \dots r_k} \varphi = S_{r_1 \dots r_k} D_{l_1 \dots l_n} \varphi, \quad (3)$$

$$1 \leq k \leq n, \quad 1 \leq r_1 < \dots < r_k \leq n, \quad 0 \leq l_j \leq k_j$$

hold.

Theorem 2 is a consequence of the lemma, since the functions $\varphi_{r_1 \dots r_k}(t)$, by the Sokhotski formulas, are linear combinations of the functions $\Phi_{p_1 \dots p_m}^-(t)$, $0 \leq m \leq n$, $1 \leq p_1 < \dots < p_m \leq n$.

3°. By H we denote the class of functions satisfying the Hölder condition with arbitrary exponents α_j , $0 < \alpha_j \leq 1$, $j = 1, 2, \dots, n$. The operators $S_{p_1 \dots p_m}$ map the class H into itself ⁽¹⁾, and $S_{p_1 \dots p_m} = S_0$. Let $\varphi(t) \in H$. We define the Cauchy transform by the equality $S\tilde{\varphi} = \varphi$; then $\tilde{\varphi}(t) \in H$, and the inversion formula has the form $S\tilde{\varphi} = \varphi$.

If $\varphi(t) \in H$ and $\psi(t) \in H$, then

$$(\varphi_{p_1 \dots p_m}(t), \psi(t)) = (-1)^m (\varphi(t), \psi_{p_1 \dots p_m}(t)), \quad (4)$$

$$(\varphi, \psi) = \int_L \varphi(\tau) \psi(\tau) d\tau. \quad (5)$$

We denote by H' the class of generalized functions ⁽²⁾ generated by the functional (5) over the basic class H . The operator $S_{p_1 \dots p_m}$ on functions of the class H' is defined by the equality (4). Let $\varphi(t) \in H'$ and $\psi(t) \in H$; then

$$(S_{p_1 \dots p_m} \varphi_{p_1 \dots p_m}, \psi) = (-1)^m (\varphi_{p_1 \dots p_m}, \psi_{p_1 \dots p_m}) = (\varphi, S_{p_1 \dots p_m}^2 \psi) = (\varphi, \psi).$$

Hence for $m = n$ the validity of the inversion formula in H' follows.

By H_k we denote the class of functions $\varphi(t)$ such that $D_{k_1 \dots k_n} \varphi(t) \in H$. The corresponding class of generalized functions will be denoted by H'_k . Using the formula for differentiating generalized functions in (3), we have

$$D_{l_1 \dots l_n} S\varphi = \frac{l_1! \dots l_n!}{(\pi i)^n} \int_L \frac{\varphi(\tau) d\tau}{\prod_{k=1}^n (\tau_k - t_k)^{l_k+1}}. \quad (6)$$

We shall say that L belongs to the class $P_{k_1 \dots k_n}(E)$, where $E \subset L$, if there exists a neighborhood $\tilde{E} \subset L$ of the manifold E , differentiable $k_j + 1$ times with respect to t_j , $j = 1, 2, \dots, n$. Suppose that on $L \in P_{k_{p_1} \dots k_{p_m}}(\Gamma_{p_1 \dots p_m})$ a surface $\Gamma_{p_1 \dots p_m}$ is given:

$$t_{p_j} = \tilde{t}_{p_j}(t_{p_1 \dots p_m}), \quad j = 1, 2, \dots, m, \quad 1 \leq m \leq n,$$

on which the delta-function $\delta(\Gamma_{p_1 \dots p_m})$ is concentrated, and let $\psi(t) \in H_k$; then, by virtue of (4)–(6), for $1 \leq l_j \leq k_{p_j}$ we have ⁽²⁾

$$\left(S \frac{\partial^{l_1 + \dots + l_m}}{\partial \tau_{p_1}^{l_1} \dots \partial \tau_{p_m}^{l_m}} \delta(\Gamma_{p_1 \dots p_m}), \psi(t) \right) = (-1)^{n + \sum_{\alpha=1}^m l_{\alpha}} \left(\delta(\Gamma_{p_1 \dots p_m}), \frac{\partial^{l_1 + \dots + l_m} \tilde{\psi}(t)}{\partial \tau_{p_1}^{l_1} \dots \partial \tau_{p_m}^{l_m}} \right) =$$

* The proof of the lemma for $n = 1$ was communicated to the author by S. Ya. Al' per.

$$= (-1)^{n + \sum_{\alpha=1}^m (l_{\alpha} + p_{\alpha}) - \frac{m(m-1)}{2}} \frac{l_1! \dots l_m!}{(\pi i)^n} \int_{\Gamma_{p_1 \dots p_m}} dt_{p_1 \dots p_m} \frac{\prod_{k=1}^{n-m} (t_{q_k} - \tau_{p_k})^{-1} \psi(t) dt}{\prod_{j=1}^m [t_{p_j} - f_{p_j}(\tau_{p_1 \dots p_m})]^{l_j+1}}.$$

Hence

$$\begin{aligned} S \frac{\partial^{l_1 + \dots + l_m}}{\partial \tau_{p_1}^{l_1} \dots \partial \tau_{p_m}^{l_m}} \delta(\Gamma_{p_1 \dots p_m}) &= (-1)^{\sum_{\alpha=1}^m (l_{\alpha} + p_{\alpha}) - \frac{m(m+1)}{2}} \frac{l_1! \dots l_m!}{(\pi i)^n} \times \\ &\times \int_{\Gamma_{p_1 \dots p_m}} \frac{\prod_{k=1}^{n-m} (\tau_{q_k} - t_{q_k})^{-1} d\tau_{p_1 \dots p_m}}{\prod_{j=1}^m [t_{p_j} - f_{p_j}(\tau_{p_1 \dots p_m})]^{l_j+1}}. \end{aligned} \quad (7)$$

4°. Let $a(t) \in H$, $\varphi(t) \in H$; then the Poincaré–Bertrand formula (1) holds, which, using the operators (1), can be written as follows:

$$Sa\tilde{\varphi} = (-1)^n \tilde{S}a\varphi + a\varphi + \sum_{k=1}^n \left(\sum_{r_1=1}^n \dots \sum_{r_k=k}^n a_{r_1 \dots r_k} \varphi_{r_1 \dots r_k} \right) +$$

$$+ \sum_{m=1}^{n-1} (-1)^m \left\{ \sum_{\substack{p_1=1 \\ p_1 < \dots < p_m}}^n \dots \sum_{p_m=m}^n S_{p_1 \dots p_m} [\varphi a_{p_1 \dots p_m} + \right. \\ \left. + \sum_{j=1}^{n-m} \left(\sum_{\substack{q_1=1 \\ q_1 < \dots < q_j}}^{n-m} \dots \sum_{q_j=j}^{n-m} \varphi_{q_1 \dots q_j} a_{p_1 \dots p_m q_1 \dots q_j} \right) \right] \right\}.$$

Replacing here $a(t)$ by $\tilde{a}(t)$ and taking into account that $S_{p_1 \dots p_m} \tilde{a} = a_{p_1 \dots p_m}$, we obtain

$$S\tilde{a}\tilde{\varphi} = \frac{1}{(\pi i)^n} \int [(-1)^n a(\tau) \varphi(\tau) + a(\tau) \varphi(t) + \\ + \sum_{k=1}^n \left(\sum_{\substack{r_1=1 \\ r_1 < \dots < r_k}}^n \dots \sum_{r_k=k}^n a(t_{r_1 \dots r_k}) \varphi(t_{r_1 \dots r_k}) \right) + \\ + \sum_{m=1}^{n-1} (-1)^m \left[\sum_{\substack{p_1=1 \\ p_1 < \dots < p_m}}^n \dots \sum_{p_m=m}^n a(\tau) \varphi(t_{\tau_{p_1 \dots p_m}}) + \right. \\ \left. + \sum_{j=1}^{n-m} \left(\sum_{\substack{q_1=1 \\ q_1 < \dots < q_j}}^{n-m} \dots \sum_{q_j=j}^{n-m} a(\tau_{t_{q_1 \dots q_j}}) \varphi(t_{\tau_{p_1 \dots p_m q_1 \dots q_j}}) \right) \right] \frac{d\tau}{\prod_{k=1}^n (\tau_k - t_k)} \equiv a(t) * \varphi(t). \quad (8)$$

The expression $a * \varphi$ with respect to the Cauchy transform plays the same role as convolutions in the theory of integral transforms ⁽⁴⁾, and we shall call it the **convolution for the Cauchy transform**.

Theorem 3. If $a(t) \in H$, $\varphi(t) \in H$; $\tilde{a}(t)$, $\tilde{\varphi}(t)$ are their Cauchy transforms, then

$$S\tilde{a}\tilde{\varphi} = a * \varphi, \quad Sa * \varphi = \tilde{a}\tilde{\varphi}. \quad (9)$$

If $a(t) \in H$, $\varphi(t) \in H'$, then formulas (9) hold in the class H' .

For $a(t) \in H$, $\varphi(t) \in H$, formulas (9) are obvious. In the case $\varphi(t) \in H'$, $a(t) \in H$, $\psi(t) \in H$, we have

$$(S\tilde{a}\tilde{\varphi}, \psi) = (-1)^n (\tilde{a}\tilde{\varphi}, \tilde{\psi}) = (-1)^n (\varphi, a\tilde{\psi}) = (\varphi, Sa\psi).$$

Replacing in the last equality on the right $S\tilde{a}\tilde{\psi}$ by formula (8) and using (4), we find that

$$(S\tilde{a}\tilde{\varphi}, \psi) = (a * \varphi, \psi).$$

Hence $S\tilde{a}\tilde{\varphi} = a * \varphi$.

5°. An equation of convolution type

$$\lambda\varphi(t) + a(t) * \varphi(t) = f(t), \quad t \in L, \quad (10)$$

where $a(t) \in H$ and $f(t) \in H$ are known functions, and λ is a parameter, for $\lambda + \tilde{a}(t) \neq 0$ on L , has in the class H the unique solution:

$$\varphi(t) = \frac{1}{(\pi i)^n} \int_L \frac{\tilde{f}(\tau)}{\lambda + \tilde{a}(\tau)} \frac{d\tau}{\prod_{k=1}^n (\tau_k - t_k)}. \quad (11)$$

Applying the operator S to both sides of (10) and taking (9) into account, we find $\tilde{\varphi}(t)$. Applying the operator S to $\tilde{\varphi}(t)$, we obtain (11).

Suppose now that $\lambda + \tilde{a}(t)$ vanishes on surfaces of the form $\Gamma_{p_1 \dots p_m} \subset L$, together with its derivatives with respect to t_{p_j} of order k_{p_j} , and $L \in P_{k_{p_1} \dots k_{p_m}}(\Gamma_{p_1 \dots p_m})$.^{*} In this case the solution of $[\lambda + \tilde{a}(t)]\tilde{\varphi}(t) = 0$ in the class H' is ⁽⁵⁾ a linear combination of the delta-function $\delta(\Gamma_{p_1 \dots p_m})$ and its derivatives up to the derivative

$$\partial^{k_{p_1} + \dots + k_{p_m} - m} / \partial t_{p_1}^{k_{p_1} - 1} \dots \partial t_{p_m}^{k_{p_m} - 1}.$$

From the validity of the inversion formula in the class H' and Theorem 3 it follows that the solution of the equation

$$\lambda\varphi(t) + a(t) * \varphi(t) = 0$$

in the class H' is a linear combination of the functions standing on the right in (7).

For the solvability of the nonhomogeneous equation in this case it is necessary and sufficient that the function $\tilde{f}(t)$ vanish together with its derivatives on the same surfaces $\Gamma_{p_1 \dots p_m}$ as the function $\lambda + \tilde{a}(t)$, and moreover the order of the derivatives of the function $\tilde{f}(t)$ that vanish on these surfaces must be no lower than that of the corresponding derivatives of the function $\lambda + \tilde{a}(t)$.

Let now L_j ($j = 1, 2, \dots, n$) be the circles $|z_j| = \rho_j$, $\alpha_j = \pi/k_j^{-1}$, l_j and k_j integers ($k_j > 0$), $te^{i\alpha} = (t_1 e^{i\alpha_1}, \dots, t_n e^{i\alpha_n})$, $a(t) \in H$ and $f(t) \in H$; then, applying to the equation

$$\varphi(t) + a(te^{i\alpha}) * \varphi(te^{i\alpha}) = f(t), \quad t \in L, \quad (12)$$

the operator S , we obtain

$$\tilde{\varphi}(t) + \tilde{a}(te^{i\alpha})\tilde{\varphi}(te^{i\alpha}) = \tilde{f}(t). \quad (13)$$

Substituting in (13) successively instead of t the values $te^{i\alpha}, \dots, te^{i(k-1)\alpha}$, where k is equal to twice the least common multiple of the numbers $k_1, \dots, k_n$, and eliminating from the resulting chain of equalities the functions $\varphi(te^{ip\alpha})$ ($p = 1, 2, \dots, k-1$), we find $\tilde{\varphi}(t)$. If

$$1 - \prod_{p=1}^k \tilde{a}(te^{ip\alpha}) \neq 0$$

on L , then, applying the operator S to $\tilde{\varphi}(t)$, we find the unique solution in the class H of equation (12)

$$\varphi(t) = \frac{1}{(\pi i)^n} \int_L \frac{\tilde{f}(\tau) + \sum_{p=1}^n (-1)^p \tilde{f}(\tau e^{ip\alpha}) \prod_{q=1}^p \tilde{a}(\tau e^{iq\alpha})}{\left[1 - \prod_{p=1}^k \tilde{a}(\tau e^{ip\alpha}) \right] \prod_{j=1}^n (\tau_j - t_j)} d\tau.$$

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* If these surfaces intersect one another along manifolds of analogous structure (and only such are studied here) of lower dimension, then these manifolds are regarded as independent.

Note: Figure translations are in progress. See original paper for figures.

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