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MATHEMATICS

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Abstract

Full Text

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NEW LIMIT THEOREMS FOR SUMS OF INDEPENDENT RANDOM VARIABLES

1. Let $X_1, X_2, \dots, X_n$ be independent identically distributed random variables, with

$$E(X_i) = 0; \quad D(X_i) = \sigma^2 > 0 \quad (i = 1, 2, \dots);$$

$$Z_n = \frac{X_1 + X_2 + \dots + X_n}{\sigma\sqrt{n}}.$$

The well-known limit theorems of probability theory show that, for any fixed x ,

$$\mathbf{P}(Z_n > x) - \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt \rightarrow 0 \quad \text{as } n \rightarrow \infty;$$

this convergence to the limit is uniform on any finite interval of values of x . However, in many problems arising in such various fields as mathematical statistics ⁽¹⁾, information theory ⁽²⁾, statistical physics of polymers ⁽³⁾, rubber chemistry ⁽⁴⁾, and analytic arithmetic ⁽⁵⁾, information is required about the behavior of $\mathbf{P}(Z_n > x_n)$, when $x_n \rightarrow \infty$ together with n , with one or another dependence on n (theorems on large deviations of Z_n). In this case the limiting relation indicated above gives only the fact that $\mathbf{P}(Z_n > x_n) \rightarrow 0$ as $n \rightarrow \infty$.

2. In the present note limit theorems on large deviations will be presented. Most of them will concern zones of normal convergence (integral and local); one of them concerns integral convergence "on the whole axis" (for arbitrary values of x_n).

Let $\psi(n) \rightarrow \infty$ be any monotone function. The sequence of intervals $[0, \psi(n)]$ will be called a **zone of normal convergence** (z. n. c.) if, as $n \rightarrow \infty$,

$$\frac{\mathbf{P}(Z_n > x)}{\frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2} du} \rightarrow 1 \tag{1}$$

for all $x \in [0, \psi(n)]$. The z. n. c. $[-\psi(n), 0]$ is defined analogously. The definition does not require uniformity of convergence in (1), although in all the theorems obtained this will be so. We first consider zones with $\psi(n) = n^\alpha$ ($\alpha > 0$ a constant).

Theorem 1. *In order that, for any $\alpha < \frac{1}{2}$, the zones $[0, n^\alpha]$ and $[-n^\alpha, 0]$ be z. n. c., it is necessary and sufficient that all moments of the variables X_i be normal. (Of course, the sufficiency here is trivial.)*

Now we may consider only values $\alpha < 1/2$.

Theorem 2. Let $\rho(n) \rightarrow \infty$ be an arbitrarily slowly increasing monotone function, and let $0 < \alpha < 1/2$. If $\alpha < 1/6$, then the necessary condition for the zones $[0, n^\alpha \rho(n)]$ and $[-n^\alpha \rho(n), 0]$ to both be zones of normal convergence, is

$$E \exp |X_j|^{\frac{4\alpha}{2\alpha+1}} < \infty. \quad (2)$$

This condition is also sufficient in order that the zones $[0, n^\alpha/\rho(n)]$, $[-n^\alpha/\rho(n), 0]$ be zones of normal convergence, and the convergence is then uniform.

If $1/6 \leq \alpha < 1/2$, consider the sequence of “critical numbers”

$$\frac{1}{6}, \frac{1}{4}, \frac{3}{10}, \dots, \frac{1}{2} \frac{s+1}{s+3}, \dots \rightarrow \frac{1}{2}. \quad (3)$$

Let

$$\frac{1}{2} \frac{s+1}{s+3} \leq \alpha < \frac{1}{2} \frac{s+2}{s+4}.$$

If the zones $[0, n^\alpha \rho(n)]$, $[-n^\alpha \rho(n), 0]$ are zones of normal convergence, condition (2) must hold and, moreover, all moments of X_j up to and including the $(s+3)$ -rd must coincide with the moments of the normal law. These two conditions are also sufficient in order that the zones $[0, n^\alpha/\rho(n)]$, $[-n^\alpha/\rho(n), 0]$ be zones of normal convergence.

Obviously, Theorem 1 follows from Theorem 2.

We now consider “narrow zones” $\psi(n) = o(n^{1/6})$. Among the theorems pertaining to this case we shall give only one, the most characteristic.

Let $h(x)$ be a monotone function subject to the following conditions:

$$(\ln \alpha)^{2+\zeta_0} \leq h(x) \leq x^{1/2} \quad (x \geq 1); \quad \zeta_0 > 0; \quad (4)$$

$$h(x) = \exp(H(\ln x)),$$

where $H(z)$ is monotone, differentiable, and

$$H'(z) \leq 1; \quad H'(z) \exp H(z) \rightarrow \infty \quad \text{as } z \rightarrow \infty. \quad (5)$$

Define a new function $\Lambda(n)$ by the equation

$$h(\sqrt{n} \Lambda(n)) = (\Lambda(n))^2. \quad (6)$$

Theorem 3. The condition

$$E \exp h(|X_j|) < \infty \quad (7)$$

is necessary in order that the zones $[0, \Lambda(n)\rho(n)]$ and $[-\Lambda(n)\rho(n), 0]$ be zones of normal convergence, and sufficient in order that the zones $[0, \Lambda(n)/\rho(n)]$, $[-\Lambda(n)/\rho(n), 0]$ be zones of normal convergence; in this case the convergence is uniform.

3. We now consider local limit theorems for the class (d) of all random variables having a continuous bounded density $g(x)$. Then Z_n will have probability density $p_{z_n}(x)$. The zone $[0, \psi(n)]$ will be called a **zone of uniform local normal convergence** (z. u. l. n. c.) if

$$\frac{p_{z_n}(x)}{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}} \rightarrow 1 \quad (8)$$

as $n \rightarrow \infty$, uniformly for $x \in [0, \psi(n)]$. The z. u. l. n. c. $[-\psi(n), 0]$ is defined analogously.

Theorem 4. For variables X_j belonging to class (d), the z. u. l. n. c. behave, with respect to necessary and sufficient

conditions indicated in Theorems 1-4, as well as necessary and sufficient conditions for the general form of random variables in Theorems 1-4.

4. Let us pass to theorems that hold "on the whole axis." We shall consider only one particular case of such theorems. Let us single out the class of all even continuous probability densities $g(x)$ such that, for $x \geq 1$,

$$\mathbf{P}(X_1 > x) = \int_x^\infty g(u) du = \frac{A_a}{x^a} + \frac{A_{a+1}}{x^{a+1}} + \dots + \frac{A_{4a+5}}{x^{4a+5}} + O\left(\frac{1}{x^{4a+5+\varepsilon}}\right), \quad (9)$$

where $\varepsilon > 0$; $a \geq 3$; a is an integer; A_j are constants. Let $X_1, X_2, \dots, X_n, \dots$ be random variables with probability density $g(x)$ from this class. Then $EX_j = 0$; let

$$D(X_j) = \sigma^2 > 0; \quad Z_n = \frac{X_1 + X_2 + \dots + X_n}{\sigma\sqrt{n}}.$$

Theorem 5. For $x \geq 1$, as $n \rightarrow \infty$, we have, uniformly in x ,

$$\frac{\mathbf{P}(Z_n > x)}{\frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2} du + r(x, \sqrt{n})} \rightarrow 1, \quad (10)$$

where $r(x, \sqrt{n})$ is a rational function of both arguments, determined by the coefficients $A_a, \dots, A_{4a+5}$ in (9). Moreover, for $x \geq n^{3/2+1/a} \ln n$ we have

$$r(x, \sqrt{n}) \sim n\mathbf{P}(X_1 > \sigma x\sqrt{n}) = n \int_{\sigma x\sqrt{n}}^\infty g(u) du. \quad (11)$$

Of course, since $g(x)$ is even, an analogous relation holds for $x \leq -1$; for $-1 < x < 1$ the well-known theorem on normal convergence holds.

An example for Theorem 5 is provided by the density $g(x) = \frac{2}{\pi} \frac{1}{(x^2 + 1)^2}$; for $x \geq 1$, $n \rightarrow \infty$, we have

$$\frac{\mathbf{P}(Z_n > x)}{\frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2} du + \frac{2}{3\pi} \frac{1}{\sqrt{n}} \frac{1}{x^3}} \rightarrow 1. \quad (12)$$

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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