

A generalization of the Phragmén-Lindelöf theorem for analytic functions to harmonic functions in space

M. A. Evgrafov, I. A. Chegis

1960

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196001.59661>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1960. Volume 134, No. 2

MATHEMATICS

M. A. EVGRAFOV and I. A. CHEGIS

A GENERALIZATION OF THE PHRAGMÉN-LINDELÖF THEOREM FROM ANALYTIC FUNCTIONS TO HARMONIC FUNCTIONS IN SPACE

(Presented by Academician M. V. Keldysh on 3 V 1960)

Theorem 1. Let $u(r, \varphi, x)$ be a harmonic function in the cylinder $r \leq a$, $0 \leq \varphi < 2\pi$, $-\infty < x < \infty$. If the conditions

$$u(a, \varphi, x) = 0, \quad \left| \frac{\partial u}{\partial r}(a, \varphi, x) \right| < C; \quad (1)$$

$$\max_{(r, \varphi)} |u(r, \varphi, x)| < C \exp e^{\pi|x|/(2+\varepsilon)a}, \quad \varepsilon > 0, \quad (2)$$

are satisfied, then $u(r, \varphi, x) \equiv 0$.

Theorem 2. Let $u(r, \theta, \varphi)$ be a harmonic function in the cone $0 < r < \infty$, $0 \leq \varphi < 2\pi$, $0 \leq \theta \leq \theta_0 < \pi$. If the conditions

$$u(r, \theta_0, \varphi) = 0, \quad \left| \frac{\partial u}{\partial \theta}(r, \theta_0, \varphi) \right| < C; \quad (1')$$

$$\max_{(\theta, \varphi)} |u(r, \theta, \varphi)| < C \exp \left(r + \frac{1}{r} \right)^{\pi/2\theta_0 - \varepsilon}, \quad \varepsilon > 0, \quad (2')$$

are satisfied, then $u(r, \theta, \varphi) \equiv 0$.

The proof of these two results is based on the following uniqueness theorem for Dirichlet series:

Theorem 3. Let

$$F(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}$$

be an entire function, and

$$|a_n|^{1/n} < \frac{C}{n^{2+\varepsilon}}, \quad \varepsilon > 0; \quad (3)$$

$$\lim_{n \rightarrow \infty} \frac{n}{\lambda_n} = \alpha, \quad 0 < \alpha < \infty, \quad \lambda_n > 0. \quad (4)$$

If, moreover, $|F(x)| < C$, $-\infty < x < \infty$, then $F(z) \equiv 0$.

We begin with the proof of Theorem 3. Put

$$F_\rho(z) = \rho \int_{-\infty}^{\infty} F(t) \exp(-e^{\rho(t-z)} + \rho(t-z)) dt. \quad (5)$$

Lemma 1. If

$$|F(t)| < C e^{-\delta|t|}, \quad -\infty < t < \infty, \quad 0 < \delta < \rho, \quad (6)$$

then $F_\rho(x + iy)$ is regular in the strip

$$-\infty < x < \infty, \quad |y| < \pi/2\rho - \eta, \quad \eta > 0, \quad (7)$$

and satisfies in this strip the inequality

$$|F_\rho(x + iy)| < C e^{-\delta|x|}. \quad (8)$$

Proof. The uniform convergence of (5) in the strip (7) follows from the estimate

$$\begin{aligned} & |F(t) \exp(-e^{\rho(t-x-iy)} + \rho(t-x-iy))| < \\ & < C \exp(-\delta|t| - e^{\rho(t-x)} \cos \rho y - \rho(t-x)), \end{aligned}$$

since there $\cos \rho y > 0$. Let us estimate $F_\rho(x + iy)$. For $x > 0$ we have

$$|F_\rho(x + iy)| < C_\rho \int_{-\infty}^{\infty} \exp(-\delta|u+x| - e^{\rho u} \cos \rho y + \rho u) du < C e^{-\delta x},$$

since $|u+x| \geq u+x$. The estimate for $x < 0$ is analogous.

Lemma 2. If

$$F(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}$$

is an entire function and conditions (3), (4) are satisfied, and $\rho > 1/(2 + \varepsilon)\alpha$, then

$$F_{\rho}(z) = \sum_{n=1}^{\infty} a_n \Gamma\left(\frac{\lambda_n}{\rho} + 1\right) e^{\lambda_n z}. \quad (9)$$

Proof. Put $z = x$ in formula (5) and make the change of variable $u = e^{\rho(t-x)}$, $t = \frac{1}{\rho} \ln u + x$. Then we obtain

$$F_{\rho}(x) = \int_0^{\infty} \left(\sum_{n=1}^{\infty} a_n e^{\lambda_n x} u^{\lambda_n/\rho} \right) e^{-u} du.$$

Applying the theorem on termwise integration (see (1), Ch. I, § 7.9), we obtain that $F_{\rho}(z)$ is representable by the Dirichlet series (9), provided this series converges. The condition for convergence of the series (9) has the form $\rho \geq 1/(2 + \varepsilon_1)\alpha$, $0 < \varepsilon_1 < \varepsilon$, since, when it is fulfilled, the coefficients of the series (9) satisfy the inequality

$$|a_n| \Gamma(\lambda_n/\rho + 1) < C \cdot n^{-\varepsilon_2 n}, \quad \varepsilon_2 > 0.$$

Lemma 3. Let the function $f(t + i\lambda)$ be regular in the strip $|\lambda| \leq \gamma$, $-\infty < t < \infty$, and satisfy there the inequality

$$|f(t + i\lambda)| < C e^{-\delta|t|}.$$

Then for the function

$$\varphi(z) = \int_{-\infty}^{\infty} f(t) e^{-tz} dt,$$

regular in the domain $|\operatorname{Re} z| < \delta$, the estimate

$$|\varphi(iy)| < C e^{-\gamma|y|}$$

holds.

Proof. Since $f(t + i\lambda) \rightarrow 0$ as $t \rightarrow \pm\infty$, $|\lambda| \leq \gamma$, we have

$$\varphi(z) = \int_{i\beta-\infty}^{i\beta+\infty} f(\zeta) e^{-z\zeta} d\zeta, \quad |\beta| \leq \gamma,$$

therefore, for $y > 0$,

$$\varphi(iy) = \int_{i\gamma-\infty}^{i\gamma+\infty} f(\zeta) e^{-iy\zeta} d\zeta = e^{-\gamma y} \int_{-\infty}^{\infty} f(t - i\gamma) e^{-ity} dt,$$

whence $|\varphi(iy)| < C e^{-\gamma y}$, $y > 0$, and analogously for $y < 0$.

Lemma 4. Denote

$$G_\rho(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right) \int_{-\infty}^{\infty} F_\rho(t) e^{-tz} dt, \quad \rho > \frac{1}{(2+\varepsilon)\alpha}. \quad (10)$$

The function $G_\rho(z)$ can be analytically continued into the half-plane $\operatorname{Re} z \geq 0$ and satisfies there the inequalities

$$|G_\rho(iy)| < C e^{\pi|y|(\alpha - \frac{1}{2\rho} + \varepsilon_3)} \quad (\varepsilon_3 > 0 \text{ arbitrary}); \quad (11)$$

$$|G_\rho(z)| < C e^{b|z|}. \quad (12)$$

Proof. Consider the functions

$$\Phi_1(z) = \int_0^{\infty} F_\rho(t) e^{-tz} dt, \quad \Phi_2(z) = \int_{-\infty}^0 F_\rho(t) e^{-tz} dt.$$

Since, by Lemma 1, $|F_\rho(t)| < C e^{-\delta|t|}$, the function $\Phi_1(z)$ is regular for $\operatorname{Re} z \geq 0$, and the integral for $\Phi_2(z)$ when $\operatorname{Re} z < \delta$ can be evaluated by termwise integration. Namely, since $\rho > 1/(2+\varepsilon)\alpha$,

$$\Phi_2(z) = \int_{-\infty}^0 \left(\sum_{n=1}^{\infty} a_n \Gamma\left(\frac{\lambda_n}{\rho} + 1\right) e^{\lambda_n t} \right) e^{-zt} dt = \sum_{n=1}^{\infty} \frac{a_n \Gamma(\lambda_n/\rho + 1)}{\lambda_n - z}. \quad (13)$$

The formula obtained gives an analytic continuation of $\Phi_2(z)$ to the whole z -plane. Thus $G_\rho(z)$ is continued into the half-plane $\operatorname{Re} z \geq 0$, and from representations (10) and (13) it is clear that $G_\rho(z)$ is regular for $\operatorname{Re} z \geq 0$.

We now prove inequality (11). Apply Lemma 3 to the function $F_\rho(t + i\lambda)$. By Lemma 1 we obtain for the parameter γ in Lemma 3 the value $\gamma = \pi/2\rho - \eta$, where $\eta > 0$ is arbitrary. Therefore

$$\left| \int_{-\infty}^{\infty} F_\rho(t) e^{-iyt} dt \right| < C e^{-(\pi/2\rho - \eta)|y|}. \quad (14)$$

For the canonical product

$$\psi(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right), \quad \lim_{n \rightarrow \infty} \frac{n}{\lambda_n} = \alpha,$$

the inequality (see, for example, (2), p. 87)

$$\max_{|z|=y} |\psi(z)| = \psi(\pm iy) \leq C e^{(\pi\alpha + \varepsilon)|y|} \quad (15)$$

holds, where $\varepsilon > 0$ is arbitrary. From (14) and (15), (11) follows.

To prove (12), note that the function $\Phi_2(z)$ is bounded on a sequence of circles $|z| = r_k$, $r_k \rightarrow \infty$, $r_{k+1} - r_k < C$, since outside the circles $|z - \lambda_n| < n^{-2}$ we have

$$|\Phi_2(z)| < C \sum_{n=1}^{\infty} n^{-\varepsilon n} n^2 < C,$$

and the sum of the diameters of the circles inside which this inequality is not valid is less than a constant. This observation and inequality (15) prove (12).

Proof of Theorem 3. Note that without loss of generality we may assume that $F(x)$ satisfies inequality (6), since otherwise we could take the function $F(z)e^{-\lambda_0 z/2}$, for which all the conditions would be fulfilled.

Now apply Lemmas 1-4 successively, taking $\rho = 1/(2 + \varepsilon_1)\alpha$. The constructed function $G_\rho(z)$ is regular in the half-plane $\operatorname{Re} z \geq 0$ and satisfies inequalities (11), (12). But from the Phragmén–Lindelöf principle it follows (see (3), Sec. III, Ch. 6, No. 327) that such a $G_\rho(z) \equiv 0$. This, in turn, means that $F_\rho(z) \equiv 0$, i.e. $a_n = 0$, $n = 1, 2, \dots$

Proof of Theorem 1. Since $u(r, \varphi, x)$ has in the cylinder $r \leq a$, $0 \leq \varphi < 2\pi$, $-\infty < x < \infty$ two continuous derivatives and satisfies the boundary condition $u(a, \varphi, x) = 0$, it can be expanded in the series

$$u(r, \varphi, x) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} e^{in\varphi} J_n\left(\lambda_{n,m} \frac{r}{a}\right) \frac{1}{J'_n(\lambda_{n,m})} (a_{n,m} e^{\lambda_{n,m} x/a} + b_{n,m} e^{-\lambda_{n,m} x/a}),$$

where $J_n(x)$ are Bessel functions; $\lambda_{n,m}$ is the m -th positive root, in order of magnitude, of $J_n(x) = 0$. Put

$$v(r, \varphi, x) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{a_{n,m}}{J'_n(\lambda_{n,m})} e^{in\varphi} J_n\left(\lambda_{n,m} \frac{r}{a}\right) e^{\lambda_{n,m} x/a}.$$

It is clear that conditions (1), (2) hold also for $v(r, \varphi, x)$, if only they hold for $u(r, \varphi, x)$. Therefore it suffices to consider $v(r, \varphi, x)$.

By virtue of the boundedness of $\frac{\partial v}{\partial r}(a, \varphi, x)$ we have

$$\left| \int_0^{2\pi} \frac{\partial v}{\partial r}(a, \varphi, x) e^{-in\varphi} d\varphi \right| < C \left| \sum_{m=1}^{\infty} a_{n,m} \lambda_{n,m} e^{\lambda_{n,m}x/a} \right| < C.$$

Consider the entire function

$$f_n(z) = \sum_{m=1}^{\infty} a_{n,m} \lambda_{n,m} e^{\lambda_{n,m}z}$$

and show that it satisfies all the conditions of Theorem 3.

First, it is well known that

$$\lim_{m \rightarrow \infty} \frac{m}{\lambda_{n,m}} = \frac{1}{\pi},$$

i.e. condition (4) is fulfilled and $\alpha = 1/\pi$. Secondly, from condition (2) of Theorem 1 we have

$$\int_0^{2\pi} \int_0^a |v(r, \varphi, x)|^2 r dr d\varphi = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} |a_{n,m}|^2 e^{2\lambda_{n,m}x/a} < C \exp 2\rho x,$$

where $\rho = \pi/(2 + \varepsilon)a$. Consequently,

$$|a_{n,m}| < C \min_x \exp \left\{ \rho x - \lambda_{n,m} \frac{x}{a} \right\} = C \exp \left[-\frac{\lambda_{n,m}}{a\rho} \left(\ln \frac{\lambda_{n,m}}{a\rho} - 1 \right) \right],$$

which gives us (for fixed n)

$$(\lambda_{n,m} |a_{n,m}|)^{1/m} < C m^{-2-\varepsilon}.$$

Thus all the conditions of Theorem 3 are satisfied, $f_n(z) \equiv 0$, and hence also $v(r, \varphi, x) \equiv 0$. The theorem is proved.

The proof of Theorem 2 is carried out in exactly the same way; only, for $u(r, \theta, \varphi)$, one uses the expansion into the series

$$u(r, \theta, \varphi) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} e^{in\varphi} H_{n,m}(\theta) (a_{n,m} r^{\lambda_{n,m}} + b_{n,m} r^{-\lambda_{n,m}-1}).$$

Here $\lambda_{n,m}$ are eigenvalues, and $H_{n,m}(\theta)$ are eigenfunctions of the problem

$$H''(\theta) + H'(\theta) \operatorname{ctg} \theta + \left(\lambda(\lambda + 1) + \frac{m^2}{\sin^2 \theta} \right) H(\theta) = 0, \quad |H(0)| < \infty, \quad H(\theta_0) = 0.$$

By refining Theorem 3 one could obtain, in Theorems 1 and 2, the replacement of conditions (2) and (2') respectively by the conditions

$$\ln \max_{(r,\varphi)} |u(r, \varphi, x)| = o(e^{\pi|x|/2a}), \quad x \rightarrow \pm\infty,$$

$$\ln \max_{(\theta,\varphi)} |u(r, \theta, \varphi)| = o\left(\left(r + \frac{1}{r}\right)^{\pi/2\theta_0}\right), \quad r \rightarrow 0, \infty.$$

A further strengthening of Theorems 1 and 2 is already impossible. In the case of Theorem 2 and $\theta_0 = \pi/2$, this is seen from the example

$$u(r, \theta, \varphi) = \cos(\mu r \sin \theta \cos \varphi) \operatorname{sh}(\mu r \cos \theta) \quad (\mu > 0 \text{ arbitrary}).$$

Theorems 1 and 2 contain the answer to certain questions posed by S. N. Mergelyan in ⁽⁴⁾.

Received
28 IV 1960

REFERENCES

- ¹ E. C. Titchmarsh, *Theory of Functions*, Moscow, 1951.
- ² M. A. Evgrafov, *Asymptotic Estimates and Entire Functions*, Moscow, 1957.
- ³ G. Pólya, G. Szegő, *Problems and Theorems in Analysis*, Moscow, 1956.
- ⁴ S. N. Mergelyan, UMN, 11, 5 (71), 3 (1956).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.