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1960

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Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1960. Volume 130, No. 6

MATHEMATICS

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THE METHOD OF MODULI IN SPACE

(Presented by Academician M. A. Lavrent'ev, XI 5, 1959)

In the geometric theory of functions of a complex variable, the notion of the modulus of topological quadrilaterals and rings is widely used. In the last decade this notion has received a new interpretation in the so-called method of extremal lengths, originating with the memoir of Ahlfors and Beurling⁽¹⁾. This interpretation makes it possible to transfer the method of moduli to Euclidean spaces of any number of dimensions; for greater clarity we shall restrict ourselves to the case of three-dimensional space.

1°. **Definition 1. Family of lines.** Let in a spatial domain D there be given some family $\{C\}$ of rectifiable curves, and also a **metric**—a measurable nonnegative function $\rho = \rho(P)$, where P is a point of D . We shall call the metric ρ **admissible** if, for every curve C of the family,

$$\int_C \rho ds \geq 1.$$

The lower bound, over all admissible ρ ,

$$M = \inf_{\rho} \int_D \rho^3 d\omega$$

will be called the **modulus** of the family $\{C\}$. A homeomorphic mapping $P_* = f(P)$ of the domain D onto D_* will be called **Q -quasiconformal** if it has continuous partial derivatives in D , its Jacobian is everywhere positive, and at each point $P \in D$ its principal linear part transforms a sphere into an ellipsoid whose ratio

$$p = \frac{a}{c}$$

of the major semiaxis to the minor one is bounded: $p(P) \leq Q$.

Theorem 1. *Under Q -quasiconformal mappings the modulus of a family of curves may change only within bounded limits depending solely on Q ; more precisely, if $\{C_*\}$ is the image of the family $\{C\}$, then*

$$M\{C\}/Q^2 \leq M\{C_*\} \leq Q^2 M\{C\}. \quad (1)$$

Indeed, let $\rho(P)$ be an arbitrary metric admissible for the family $\{C\}$. On D_* define the function

$$\rho_*(P_*) = \rho(P)/\mu(P),$$

where $P = f^{-1}(P_*)$ and $\mu = r/a$ is the minimal stretching of f at the point P (r is the radius of the sphere corresponding to the ellipsoid); ρ_* is, evidently, admissible for $\{C_*\}$, since $ds_* \geq \mu ds$ (ds_* is the element of length of C_*) and

$$\int_{C_*} \rho_* ds_* \geq \int_C \rho ds \geq 1.$$

But the Jacobian of the mapping

$$J = \frac{r^3}{abc} = \frac{\mu^3 a^2}{bc} \leq \mu^3 Q^2$$

($a \geq b \geq c$ are the semiaxes of the ellipsoid transformed into a sphere), therefore

$$M\{C_*\} \leq \inf_{\rho_*} \int_{D_*} \rho_*^3 d\omega_* = \inf_{\rho} \int_D \rho^3 \frac{J}{\mu^3} d\omega \leq Q^2 M\{C\}.$$

Applying the same reasoning to the mapping f^{-1} , also Q -quasiconformal, we obtain the left-hand side of (1).

Definition 2. Family of surfaces. Let $\{S\}$ be a family of quadrable surfaces in D ; the metric $\rho = \rho(P)$ will be con-

be admissible if for any surface S of the family

$$\int_S \rho^2 d\sigma \geq 1.$$

The **modulus** of the family $\{S\}$ will be called

$$M\{S\} = \inf_{\rho} \int_D \rho^3 d\omega,$$

where the lower bound is taken over all admissible ρ .

Theorem 2. If $\{S_*\}$ is the family corresponding to $\{S\}$ under a Q -quasiconformal mapping, then

$$M\{S\}/Q \leq M\{S_*\} \leq QM\{S\}. \quad (2)$$

From Theorems 1 and 2 it follows that the moduli of families of curves and surfaces are invariant under conformal mappings.

Remark. Using the results of A. I. Markushevich ⁽²⁾, one can substantially weaken the function-theoretic restrictions imposed above in the definition of Q -quasiconformal mappings.

2°. Example 1. D is a rectangular parallelepiped:

$$0 < x < \alpha, \quad 0 < y < \beta, \quad 0 < z < \gamma;$$

$\{C\}$ is the family of all rectifiable curves joining the upper and lower bases. We have

$$M\{C\} = \alpha\beta/\gamma^2. \quad (3)$$

Indeed, since the length of any C is not less than γ , $\rho_0 = 1/\gamma$ is an admissible metric, to which there corresponds the volume $M = \alpha\beta/\gamma^2$. Let ρ be an arbitrary admissible metric; by Fubini's theorem and Hölder's inequality,

$$\int_D \rho^3 d\omega = \int_0^\alpha dx \int_0^\beta dy \int_0^\gamma \rho^3 dz \geq \frac{1}{\gamma^2} \int_0^\alpha dx \int_0^\beta dy \left(\int_0^\gamma \rho dz \right)^3 \geq \frac{\alpha\beta}{\gamma^2},$$

for, since the rectilinear segments belong to $\{C\}$, we have

$$\int_0^\gamma \rho dz \geq 1;$$

thus (3) is proved.

Example 1'. D is as in Example 1; $\{S\}$ is the family of all quadrable surfaces whose boundaries lie on the lateral faces of D . The metric

$$\rho_0 = 1/\sqrt{\alpha\beta}$$

is admissible, and the corresponding volume is

$$M = \gamma/\sqrt{\alpha\beta}.$$

Let ρ be an arbitrary admissible metric; by Fubini's theorem and Hölder's inequality,

$$\int_D \rho^3 d\omega = \int_0^\gamma dz \iint_{S_z} \rho^3 d\sigma \geq \frac{1}{\sqrt{\alpha\beta}} \int_0^\gamma \left(\iint_{S_z} \rho^2 d\sigma \right)^{3/2} dz,$$

where S_z is the section of D at height z . Since $S_z \in \{S\}$, we have

$$\iint_{S_z} \rho^2 d\sigma \geq 1$$

and

$$\int_D \rho^3 d\omega \geq \frac{\gamma}{\sqrt{\alpha\beta}}.$$

Thus:

$$M\{S\} = \gamma/\sqrt{\alpha\beta}. \quad (3')$$

Remark. Let $\{C_1\}$, $\{C_2\}$, and $\{C_3\}$ be the families of rectifiable curves joining the corresponding opposite faces of D . From (3) we have

$$M\{C_1\}M\{C_2\}M\{C_3\} = 1.$$

If $\{S_k\}$ is the family of quadrable surfaces ending on the faces free from the ends of $\{C_k\}$, then from (3) and (3') we obtain

$$M\{C_k\} = 1/M^2\{S_k\}.$$

Example 2. D is a spherical sector:

$$r_1 < r < r_2, \quad 0 \leq \varphi < 2\pi, \quad 0 < \theta < \theta_0$$

(r, φ, θ are polar coordinates); $\{C\}$ is the family of all rectifiable curves connecting the spherical boundaries of D . We have

$$M\{C\} = 4\pi \sin^2(\theta_0/2) / \ln^2(r_2/r_1). \quad (4)$$

Indeed, the metric $\rho_0 = 1/r \ln(r_2/r_1)$ is admissible, since

$$\int_C \rho_0 ds \geq \int_{r_1}^{r_2} \rho_0 dr = 1$$

for any C , and the corresponding volume is equal to the right-hand side of (4). Let ρ be an arbitrary admissible metric,

$$\int_D \rho^3 d\omega = \int_0^{2\pi} d\varphi \int_0^{\theta_0} \sin \theta d\theta \int_{r_1}^{r_2} \rho^3 r^2 dr;$$

but since, if

$$\int_{r_1}^{r_2} \rho dr \geq 1,$$

then

$$\int_{r_1}^{r_2} \rho^3 r^2 dr \geq \frac{1}{\ln^2(r_2/r_1)}$$

(this is proved by the methods of the calculus of variations), it follows that

$$\int_D \rho^3 d\omega \geq \frac{1}{\ln^2(r_2/r_1)} \int_0^{2\pi} d\varphi \int_0^{\theta_0} \sin \theta d\theta = \frac{4\pi \sin^2(\theta_0/2)}{\ln^2(r_2/r_1)},$$

as required.

Example 2'. D as in Example 2; $\{S\}$ is the family of all quadrable surfaces whose boundaries lie on the conical part of the boundary of D . We have

$$M\{S\} = \ln(r_2/r_1)/2\sqrt{\pi} \sin(\theta_0/2). \quad (4')$$

This is proved by methods analogous to the preceding ones, using Hölder's inequality.

Remark. From (4) and (4') it follows that $M\{C\} = 1/M^2\{S\}$.

Example 3. D is a cylindrical ring: $r_1 < r < r_2$, $0 \leq \varphi < 2\pi$, $0 < z < H$ (r, φ, z are cylindrical coordinates); $\{C\}$ is the family of all rectifiable curves connecting the upper and lower bases of D ; $\{S\}$ is the family of all quadrable surfaces homotopic in D to the rings $r_1 < r < r_2$, $z = z_0$ ($0 < z_0 < H$). We have

$$M\{C\} = 1/M^2\{S\} = \pi(r_2^2 - r_1^2)/H^2. \quad (5)$$

Example 4. D as in Example 3; $\{C\}$ is the family of all closed rectifiable curves homotopic to the circles $r = r_0$, $z = z_0$ ($r_1 < r_0 < r_2$, $0 < z_0 < H$); $\{S\}$ is the family of all quadrable surfaces homotopic to the rectangles $\varphi = \varphi_0$, $r_1 < r < r_2$, $0 < z < H$. We have

$$M\{C\} = 1/M^2\{S\} = H(r_2 - r_1)/4\pi^2 r_1 r_2. \quad (6)$$

Example 5. D as in Example 3; $\{C\}$ is the family of all rectifiable curves homotopic to the segments $\varphi = \varphi_0$, $z = z_0$, $r_1 < r < r_2$ ($0 < z_0 < H$); $\{S\}$ is the family of all quadrable surfaces homotopic to the cylinders $r = r_0$, $0 < z < H$ ($r_1 < r_0 < r_2$). We have

$$M\{C\} = 1/M^2\{S\} = \pi H/2(\sqrt{r_2} - \sqrt{r_1})^2. \quad (7)$$

3°. General properties. We shall call an admissible metric **extremal** if, in it, the volume of the domain coincides with the modulus of the family of curves or surfaces under consideration. The following theorem asserts the uniqueness of the extremal metric for a given family.

Theorem 3. *The extremal metric of a given family of curves or surfaces is determined uniquely, up to a set of volume measure 0.*

We shall prove the theorem for the case of a family of curves $\{C\}$. Let ρ_1 and ρ_2 be two extremal metrics for this family, i.e.

$$\int_D \rho_1^3 d\omega = \int_D \rho_2^3 d\omega = M\{C\}.$$

The metric $\rho = (\rho_1 + \rho_2)/2$ is obviously admissible for this family and, integrating over D the identity

$$\frac{1}{2}(\rho_1^3 + \rho_2^3) = \frac{1}{8}(\rho_1 + \rho_2)^3 + \frac{3}{8}(\rho_1 - \rho_2)^2(\rho_1 + \rho_2),$$

we find

$$\int_D (\rho_1 - \rho_2)^2(\rho_1 + \rho_2) d\omega \leq 0,$$

whence $\rho_1 = \rho_2$ almost everywhere in D . For the case of a family of surfaces the proof proceeds similarly, if one chooses the admissible metric

$$\rho = \sqrt{\frac{1}{2}(\rho_1^2 + \rho_2^2)}$$

and uses the identity

$$\frac{1}{2}(\rho_1^3 + \rho_2^3) = \rho^3 + \frac{1}{4}(\rho_1 - \rho_2)^2 \frac{\rho_1^4 + 4\rho_1\rho_2\rho^2 + \rho_2^4}{\rho_1^3 + 2\rho^3 + \rho_2^3}.$$

The following two theorems extend the well-known Grötzsch principles from the theory of conformal mappings (see, for example, (3)).

Theorem 4. *If two families of curves (or two families of surfaces) with moduli M_1, M_2 lie in disjoint domains D_1, D_2 , and a third family, lying in $D = D_1 \cup D_2$, is obtained by the union of the given two, then the modulus of the latter satisfies*

$$M \geq M_1 + M_2. \quad (8)$$

Let ρ be an admissible metric for the third family; then $\rho_k = \rho$ in D , $\rho_k = 0$ in $D \setminus D_k$, will obviously be admissible for the k -th family ($k = 1, 2$), and (8) follows from the identity

$$\int_D \rho^3 d\omega = \int_{D_1} \rho_1^3 d\omega + \int_{D_2} \rho_2^3 d\omega.$$

Corollary. Under an “extension” of a family (of surfaces or curves), its modulus can only increase.

Theorem 5. *If the families of curves $\{C_1\}, \{C_2\}$ are situated in disjoint domains D_1, D_2 , and each curve of the family $\{C\}$ consists of one C_1 and one C_2 , then*

$$\frac{1}{\sqrt{M\{C\}}} \geq \frac{1}{\sqrt{M\{C_1\}}} + \frac{1}{\sqrt{M\{C_2\}}}. \quad (9)$$

For families of surfaces $\{S_1\}, \{S_2\}$ and $\{S\}$ having the same properties,

$$\frac{1}{M^2\{S\}} \geq \frac{1}{M^2\{S_1\}} + \frac{1}{M^2\{S_2\}}. \quad (10)$$

Let ρ_k be metrics admissible for the families of curves $\{C_k\}$; then for any λ , $0 < \lambda < 1$, the metric $\rho = \lambda\rho_1$ in D_1 , $\rho = (1 - \lambda)\rho_2$ in D_2 , is obviously admissible for $\{C\}$, and by integration we obtain

$$M\{C\} \leq \lambda^3 M\{C_1\} + (1 - \lambda)^3 M\{C_2\}.$$

Choosing

$$\lambda = \frac{\sqrt{M\{C_2\}}}{\sqrt{M\{C_1\}} + \sqrt{M\{C_2\}}},$$

we obtain (9). For families of surfaces the proof is analogous; one need only put

$$\rho = \sqrt{\lambda}\rho_1 \quad \text{in } D_1, \quad \rho = \sqrt{1 - \lambda}\rho_2 \quad \text{in } D_2,$$

and then choose

$$\lambda = \frac{M^2\{S_2\}}{M^2\{S_1\} + M^2\{S_2\}}.$$

Corollary. Under a “lengthening” of a family (of curves or surfaces), its modulus can only decrease.

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Received
30 X 1959

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Note: Figure translations are in progress. See original paper for figures.

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